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The character spectrum of $\beta(\mathbb{N})$

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ABSTRACT

We show the consistency of: the set of regular cardinals which are the character of some ultrafilter on $\mathbb{N}$ can be quite chaotic, in particular can have many gaps.

0. Introduction

The set of characters of non-principal ultrafilters on $\mathbb{N}$, that we call the character spectrum and denote by Sp_χ, is naturally of interest to topologists and set theorists alike, see Definition 0.1 below. A natural question is what can this set of cardinals be? The first result on Sp_χ is Pospíšil's proof that $\mathfrak{c} \in \mathrm{Sp}_\chi$.

It is consistent that $\mathrm{Sp}_\chi = \{2^{\aleph_0}\}$, since Martin's Axiom implies $\mathrm{Sp}_\chi = \{2^{\aleph_0}\}$. Nevertheless, $\mathrm{Sp}_\chi = \{2^{\aleph_0}\}$ is not a theorem of ZFC. Juhász (see [1]) proved the consistency of the existence of a non-principal ultrafilter D so that $\chi(D) < 2^{\aleph_0}$. Kunen (in [2]) mentions that $\aleph_1 \in \mathrm{Sp}_\chi$ in the side-by-side Sacks model.

Those initial results show that $\chi(D)$ is not a trivial cardinal invariant. But we may wonder whether Sp_χ is an interesting set. For instance, can Sp_χ include more than two members? Does it have to be a convex set? It is proved in [3, §6] that $|\mathrm{Sp}_\chi|$ large is consistent, e.g. $2^{\aleph_0}$ is large and all regular uncountable $\kappa \leqslant 2^{\aleph_0}$ (or just of uncountable cofinality) belong to it. It was asked there: among regular cardinals is it convex? Now (proved in [6]) Sp_χ does not have to be convex. In the model of [6], there is a triple of cardinals (μ, κ, λ) such that $\mu < \kappa < \lambda$, $\mu, \lambda \in \mathrm{Sp}_\chi$ but $\kappa \notin \mathrm{Sp}_\chi$. In the present paper we show that Sp_χ may exhibit much more chaotic behavior.

To be specific, starting from two disjoint sets Θ_1 and Θ_2 of regular uncountable cardinals we produce a forcing notion $\mathbb{P}$ which forces the following properties:

(a) no cardinal (of $\mathbf{V}$) is collapsed in $\mathbf{V}^{\mathbb{P}}$;
(b) $2^{\aleph_0}$ is an upper bound for the union of Θ_1 and Θ_2;
(c) $\Theta_1 \subseteq \mathrm{Sp}_\chi$ whereas $\Theta_2 \cap \mathrm{Sp}_\chi = \emptyset$.

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The proof requires that each element of Θ_2 be measurable and that $\theta \in \Theta_1$ satisfy $\theta^{<\theta} = \theta$. This means that in the extension all members of Θ_2 are weakly inaccessible and hence that we also do not know for certain that there are successor cardinals outside Sp_χ.

In the last section we show that we can, e.g. specify $\mathrm{Sp}_\chi \cap \aleph_\omega$, basically at will: if we have infinitely many measurable cardinals then we can make the intersection be $\{\aleph_n\colon n \in u\}$ for any subset of $[1, \omega)$ that has no large gaps, i.e. for every n at least one of n and $n+1$ belongs to u. If we assume infinitely many compact cardinals then we can realize any ground model subset of $[1, \omega)$, e.g. $\mathrm{Sp}_\chi \cap \aleph_\omega$ can be even $\{\aleph_p\colon p \text{ prime}\}$.

Let us try to explain how do we do this. A purpose of [3] is to create a large Sp_χ. It provides a way to ensure many cardinals are in Sp_χ. On the other hand, [6] provides a way for guaranteeing a cardinal is not in Sp_χ. Here we try to combine the methods, hence creating a large set with many prescribed gaps which establishes Sp_χ in $\mathbf{V}^{\mathbb{P}}$.

For adding cardinals we use systems of filters, so we deal with them and with the "one step forcing" in Section 1; we use such systems indexed, e.g. by κ-trees, and in the end force by a suitable product of those trees, not adding reals. In this direction we do not need large cardinal assumptions. For eliminating cardinals we need, essentially, measurables in the ground model. After the forcing with $\mathbb{P}$, our measurable cardinals become weakly inaccessible, and we show that they do not belong to Sp_χ.

We emphasize that for adding a cardinal to $\Theta_1 \subseteq \mathrm{Sp}_\chi$, we have to assume $\theta = \theta^{<\theta}$. Moreover, Θ_2 consists (in the ground model) of measurable cardinals which remain weakly inaccessible (= regular limit) cardinals in $\mathbf{V}^{\mathbb{P}}$. Consequently, in Section 2 we do not know for certain that there are successor cardinals outside Sp_χ. As in many other cases, to deal with "small, e.g. successor" cardinals we have also to collapse.

The last section of the paper is devoted to the set $\mathrm{Sp}_\chi \cap \aleph_\omega$. Let $u \subseteq \omega$ be any set (e.g., $u = \{p\colon p \text{ is a prime number}\}$). If we assume that there are infinitely many compact cardinals in the ground model, then we can force $\mathrm{Sp}_\chi \cap \aleph_\omega = \{\aleph_n\colon n \in u\}$. Assuming just the existence of infinitely many measurable cardinals, we can prove a similar result with some restrictions on u. We need that $|u \cap \{n, n+1\}| \geqslant 1$ for every $n \in \omega$.

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Recall

Definition 0.1. 1) For an ultrafilter D on $\mathbb{N}$ let $\chi(D)$, the character of D be $\min\{|\mathscr{A}|\colon \mathscr{A} \subseteq D$ and every member of D include some member of $\mathscr{A}\}$.

2) The character spectrum of non-principal ultrafilters on $\mathbb{N}$ is $\mathrm{Sp}_\chi := \{\chi(D)\colon D \text{ a non-principal ultrafilter on } \mathbb{N}\}$.

1. Preliminaries

This section is devoted to definitions and facts, needed for proving the main results of the paper. We present filter systems $\bar{D} = \langle D_t\colon t \in I\rangle$ and we deal with the one step forcing $\mathbb{Q}_{\bar{D}}$ where $\bar{D} = \langle D_\eta\colon \eta \in {}^{\omega>}\omega\rangle$, D_η a filter on $\mathbb{N}$ containing the co-finite subsets of $\mathbb{N}$; when $\mathbb{P}_1 * \mathbb{Q}_{\underset{\sim}{\bar{D}}_1} \lessdot \mathbb{P}_2 * \mathbb{Q}_{\underset{\sim}{\bar{D}}_2}$, and with frames $\mathbf{d} = (\bar{D}_{\mathbf{d}}, F_{\mathbf{d}})$ for analyzing $\mathbb{Q}_{\mathbf{d}}$-names of $\underset{\sim}{A}$ of subsets of $\mathbb{N}$ modulo the filter on $\mathbb{N}$ which $F_{\mathbf{d}}$ generated, in particular, a derived $\mathbb{Q}_{\bar{D}}$-name of an ideal $\underset{\sim}{\mathrm{id}}_{\mathbf{d}}$.

Definition 1.1. For forcing notion $\mathbb{P}_1, \mathbb{P}_2$ (i.e. quasi-orders).

1) $\mathbb{P}_1 \subseteq \mathbb{P}_2$ <u>iff</u> $p \in \mathbb{P}_1 \Rightarrow p \in \mathbb{P}_2$ and for every $p, q \in \mathbb{P}_1$ we have $\mathbb{P}_1 \models$ "$p \leqslant q$" iff $\mathbb{P}_2 \models$ "$p \leqslant q$".

2) $\mathbb{P}_1 \subseteq_{\mathrm{ic}} \mathbb{P}_2$ <u>iff</u> $\mathbb{P}_1 \subseteq \mathbb{P}_2$ and for every $p, q \in \mathbb{P}_1$ we have p, q are compatible in $\mathbb{P}_1$ iff p, q are compatible in $\mathbb{P}_2$.

3) $\mathbb{P}_1 \lessdot \mathbb{P}_2$ <u>iff</u>:

$\boxplus_1$ $\mathbb{P}_1 \subseteq \mathbb{P}_2$ and every maximal antichain of $\mathbb{P}_1$ is a maximal antichain of $\mathbb{P}_2$,

equivalently

$\boxplus_2$ $\mathbb{P}_1 \subseteq_{\mathrm{ic}} \mathbb{P}_2$ and for every $p_2 \in \mathbb{P}_2$ for some $p_1 \in \mathbb{P}_1$ we have $p_1 \leqslant_{\mathbb{P}_1} p \Rightarrow (p_2, p$ are compatible in $\mathbb{P}_2)$.

Definition/Observation 1.2. 1) For $\mathscr{A} \subseteq \mathscr{P}(\mathbb{N})$ let $\mathrm{fil}(\mathscr{A}) = \{B \subseteq \omega\colon \bigcap_{\ell<n} A_\ell \subseteq^* B$ for some $n < \omega$ and $A_0, \ldots, A_{n-1} \in \mathscr{A}\}$; so if $\mathscr{A}$ is empty then $\mathrm{fil}(\mathscr{A})$ is the filter of co-finite sets. We may forget to distinguish between $\mathscr{A}$ and $\mathrm{fil}(\mathscr{A})$.

2) $\mathrm{fil}(\mathscr{A})$ is a filter on $\mathbb{N}$ extending the filter of co-bounded subsets of $\mathbb{N}$ but possibly $\mathrm{fil}(\mathscr{A}) = \mathscr{P}(\mathbb{N})$, equivalently $\emptyset \in \mathrm{fil}(\mathscr{A})$.

3) For a filter D on X let $D^+ = \{Y \subseteq X\colon Y \neq \emptyset \bmod D\}$.

Definition 1.3. Let I be a partial order or just a quasi-order.

1) We say $\bar{D}$ is an I-filter system <u>when</u>:

(a) $\bar{D} = \langle D_t\colon t \in I\rangle$
(b) $D_t \subseteq \mathscr{P}(\mathbb{N})$ but $\emptyset \notin \mathrm{fil}(D_t)$
(c) if $s \leqslant_I t$ then $\mathrm{fil}(D_s) \subseteq \mathrm{fil}(D_t)$.

2) We say $\bar{D}$ is an ultra I-filter system when in addition:

(d) if $s \in I$, $A \subseteq \mathbb{N}$ and $A \neq \emptyset$ mod D_s <u>then</u> for some t we have $s \leqslant_I t$ and $A \in \mathrm{fil}(D_t)$.

3) If $\bar{D}_\ell$ is an I_ℓ-filter system for $\ell = 1, 2$ <u>then</u> we let ($\bar{D}_\ell = \langle D_{\ell,t}\colon t \in I_\ell\rangle$ and):

(a) $\bar{D}_1 \leqslant \bar{D}_2$ <u>means</u> $I_1 \subseteq I_2$ (as quasi-orders, so possibly $I_1 = I_2$) and $s \in I_1 \Rightarrow D_{1,s} \subseteq D_{2,s}$
(b) $\bar{D}_1 \leqslant^* \bar{D}_2$ means $I_1 \subseteq I_2$ and $s \in I_1 \Rightarrow \mathrm{fil}(D_{1,s}) \subseteq \mathrm{fil}(D_{2,s})$
(c) $\bar{D}^1 \leqslant^{\odot} \bar{D}^2$ means $I_1 \subseteq I_2$ and $s \in I_1 \Rightarrow \mathrm{fil}(D_{1,s}) = \mathrm{fil}(D_{2,s})$
(d) $\bar{D}^1 =^* \bar{D}^2$ means $I_1 = I_2$ and $s \in I_1 \Rightarrow \mathrm{fil}(D_{1,s}) = \mathrm{fil}(D_{2,s})$.

Observation 1.4. Let I be a partial order.

0) $\leqslant$, $\leqslant^{\odot}$ and $\leqslant^*$ quasi-order the set of I-filter systems and $\langle \mathrm{fil}(D_t)\colon t \in I\rangle$ is an I-filter system for any I-filter system $\bar{D}$ and $\bar{D}_1 \leqslant \bar{D}_2 \Rightarrow \bar{D}_1 \leqslant^* \bar{D}_2$ and $\bar{D}^1 =^* \bar{D}^2 \Rightarrow \bar{D}^1 \leqslant^{\odot} \bar{D}^2 \Rightarrow \bar{D}^1 \leqslant^* \bar{D}^2$ and $\bar{D}_1 \leqslant \bar{D}_2 \leqslant D_1 \Rightarrow \bar{D}_1 = \bar{D}_2$ and $\bar{D}_1 \leqslant^* \bar{D}_2 \leqslant^* \bar{D}_1 \Rightarrow \bar{D}_1 =^* \bar{D}_2$.

1) If $A_s \in [\mathbb{N}]^{\aleph_0}$ for each $s \in I$ and $A_t \subseteq^* A_s$ for $s \leqslant_I t$ <u>then</u> there is an I-filter system $\bar{D}$ such that $s \in I \Rightarrow D_s = \{A_s\}$.

2) If $\bar{D}$ is an I-filter system <u>then</u> for some ultra I-filter system $\bar{D}'$ we have $\bar{D} \leqslant \bar{D}'$.

3) If $\bar{D}$ is an I-filter system, $s \in I$ and $A \subseteq \omega$ and $(\forall t)[s \leqslant_I t \Rightarrow A \neq \emptyset \text{ mod } \mathrm{fil}(D_t)]$, <u>then</u> for some I-filter system $\bar{D}'$ we have $\bar{D} \leqslant \bar{D}'$ and $A \in D'_s$.

4) If $\langle \bar{D}_\alpha\colon \alpha < \delta\rangle$ is an $\leqslant$-increasing sequence of I-filter systems <u>then</u> some I-filter system $\bar{D}_\delta$ is an upper bound of the sequence; in fact, one can use the limit, i.e. $D_{\delta,s} = \bigcup\{D_{\alpha,s}\colon \alpha < \delta\}$; similarly for $\leqslant^*$-increasing.

5) If $\bar{D}$ is an I-filter system and $\bar{D}' = \langle \mathrm{fil}(D_t)\colon t \in I\rangle$ <u>then</u> $\bar{D} \leqslant \bar{D}'$.

6) If $\bar{D}$ is an I-filter system and each D_t is an ultrafilter on ω <u>then</u> $\bar{D}$ is an ultra I-filter system and necessarily $s \leqslant_I t \Rightarrow D_s = D_t$.

7) If $\bar{D}_1$ is an ultra I-filter system and $\bar{D}_2$ is an I-filter system such that $\bar{D}_1 \leqslant^* \bar{D}_2$ <u>then</u> $\bar{D}_1 \leqslant^{\odot} \bar{D}_2$.

8) Assume $\mathbb{P}_1 \lessdot \mathbb{P}_2$ and $\Vdash_{\mathbb{P}_1}$ "$\underset{\sim}{\bar{D}}_\ell$ is an I-filter system" for $\ell = 1, 2$. If $\Vdash_{\mathbb{P}_1}$ "$\underset{\sim}{\bar{D}}_1 \leqslant \underset{\sim}{\bar{D}}_2$" <u>then</u> $\Vdash_{\mathbb{P}_2}$ "$\underset{\sim}{\bar{D}}_1 \leqslant \underset{\sim}{\bar{D}}_2$"; also if $\Vdash_{\mathbb{P}_1}$ "$\underset{\sim}{\bar{D}}_1 \leqslant^* \underset{\sim}{\bar{D}}_2$ <u>then</u> $\Vdash_{\mathbb{P}_2}$ "$\underset{\sim}{\bar{D}}_1 \leqslant^* \underset{\sim}{\bar{D}}_2$".

9) If $\mathbb{P}_1 \lessdot \mathbb{P}_2$ and $\Vdash_{\mathbb{P}_\ell}$ "$\underset{\sim}{\bar{D}}_\ell$ is an $\underset{\sim}{I}_\ell$-filter system" for $\ell = 1, 2$ and $\Vdash_{\mathbb{P}_1}$ "$\underset{\sim}{\bar{D}}_1$ is ultra" and $\Vdash_{\mathbb{P}_2}$ "$\underset{\sim}{\bar{D}}_1 \leqslant^* \underset{\sim}{\bar{D}}_2$" <u>then</u> $\Vdash_{\mathbb{P}_2}$ "$\underset{\sim}{D}_{1,t} \subseteq \underset{\sim}{D}_{2,t}$ and $(\mathrm{fil}(\underset{\sim}{D}_{1,t})^+)^{\mathbf{V}[\mathbb{P}_1]} \subseteq \mathrm{fil}(\underset{\sim}{D}_{2,t})^+$".

Proof. 0) Easy.

1) Check.

2) Use parts (3), (4), easy, but we elaborate. We try to choose $\bar{D}_\alpha$ by induction on $\alpha < (2^{\aleph_0} + |I|)^+$ such that $\bar{D}_\alpha$ is an I-filter system, $\beta < \alpha \Rightarrow \bar{D}_\beta \leqslant \bar{D}_\alpha$ and for each $\alpha = \beta + 1$ for some t, $D_{\alpha,t} \neq D_{\beta,t}$. For $\alpha = 0$ let $\bar{D}_\alpha = \bar{D}$, for α limit use part (4) and for $\alpha = \beta + 1$ if $\bar{D}_\beta$ is not ultra, use part (3). By cardinality consideration for some β, $\bar{D}_\beta$ is defined but we cannot define $\bar{D}_{\beta+1}$ so necessarily $\bar{D}_\beta$ is ultra as required.

3)–9) Easy, too. $\square_{1.4}$

Claim 1.5. 1) *Assume the quasi-order I as a forcing notion adds no new reals. An I-filter system $\bar{D}$ is ultra <u>iff</u> $\Vdash_I$ "$\bigcup\{\mathrm{fil}(D_t)\colon t \in \underset{\sim}{\mathbf{G}}_I\}$ is an ultrafilter on ω".*

2) *Assume the quasi-order I as a forcing notion adds no new ω_1-sequences of ordinals and $\mathbb{P}$ is a* c.c.c. *forcing notion (or just I is $\aleph_1$-complete). If $\Vdash_{\mathbb{P}}$ "$\langle \underset{\sim}{D}_t\colon t \in I\rangle$ is an I-filter system" <u>then</u> $\Vdash_{\mathbb{P}} \Vdash_I$ "$\bigcup\{\mathrm{fil}(\underset{\sim}{D}_t)\colon t \in \underset{\sim}{G}_I\}$ is an ultrafilter on $\mathbb{N}$" iff $\Vdash_{\mathbb{P}}$ "$\langle \underset{\sim}{D}_t\colon t \in I\rangle$ is an ultra I-filter system".*

Proof. Easy. $\square_{1.5}$

Discussion 1.6. An I-filter system $\bar{D}$ may be "degenerated", i.e. $D_t = D$ is an ultrafilter, the same for every $t \in I$. But in this case adding a generic set to I will not add naturally a new ultrafilter, which is our aim here.

Definition 1.7. 1) For $\bar{D} = \langle D_\eta\colon \eta \in {}^{\omega>}\omega\rangle$, each D_η a filter on $\mathbb{N}$ let $\mathbb{Q}_{\bar{D}}$ be

$$\begin{aligned}\big\{T\colon\ & T \subseteq {}^{\omega>}\omega \text{ is closed under initial segments, and for some}\\ & \mathrm{tr}(T) \in {}^{\omega>}\omega,\ \text{the trunk of } T,\ \text{we have:}\\ & \text{(i)}\quad \ell \leqslant \ell g\big(\mathrm{tr}(T)\big) \Rightarrow T \cap {}^{\ell}\omega = \big\{\mathrm{tr}(T) \restriction \ell\big\}\\ & \text{(ii)}\quad \mathrm{tr}(T) \trianglelefteq \eta \in {}^{\omega>}\omega \Rightarrow \big\{n\colon \eta^\frown\langle n\rangle \in T\big\} \in \underset{\sim}{D}_\eta\big\}\end{aligned}$$

ordered by inverse inclusion.

2) For $p \in \mathbb{Q}_{\bar{D}}$ let $\mathrm{wfst}(p, \bar{D})$ be the set of pairs (S, ζ) such that:

(a) (α) $S \subseteq \{\eta \in p\colon \mathrm{tr}(p) \trianglelefteq \eta \in p\}$
  (β) $\mathrm{tr}(p) \in S$
  (γ) $\mathrm{tr}(p) \trianglelefteq \nu \triangleleft \eta \in S \Rightarrow \nu \in S$
(b) (α) ζ is a function from S into ω_1
  (β) if $\nu \triangleleft \eta$ are from S then $\zeta(\nu) > \zeta(\eta)$
  (γ) if $\eta \in S$ and $\zeta(\eta) > 0$ <u>then</u> $\{k\colon \eta^\frown\langle k\rangle \in S\} \neq \emptyset \bmod D_\eta$.

3) If $p \in \mathbb{Q}_{\bar{D}}$ and $\nu \in p$ then we let $p^{[\nu]} = \{\rho \in p\colon \rho \trianglelefteq \nu \text{ or } \nu \trianglelefteq \rho\}$.

4) If $\bar{D} = \langle D_\eta\colon \eta \in {}^{\omega>}\omega\rangle$, $D_\eta = D$ for $\eta \in {}^{\omega>}\omega$ then let $\mathbb{Q}_D = \mathbb{Q}_{\bar{D}}$ and $\mathrm{wfst}(p, D) = \mathrm{wfst}(p, \bar{D})$; we may write η instead of p when this holds for some $p \in \mathbb{Q}_{\bar{D}}$ with $\mathrm{tr}(p) = \eta$; wfst stands for well founded sub-tree.

Claim 1.8. *Assume $\eta^* \in {}^{\omega>}\omega$, D_η is a filter on $\mathbb{N}$ for $\eta \in {}^{\omega>}\omega$ and $\mathscr{Y}$ is a subset of $\Lambda = \Lambda_{\eta^*} = \{\eta\colon \eta^* \trianglelefteq \eta \in {}^{\omega>}\omega\}$. <u>Then</u> exactly one of the following clauses holds*:

(a) *there is $q \in \mathbb{Q}_{\bar{D}}$ such that*
  (α) $\eta^* = \mathrm{tr}(q)$
  (β) *$\mathscr{Y} \cap q = \emptyset$, equivalently $q^+ = q \setminus \{\mathrm{tr}(q) \restriction \ell\colon \ell < \ell g(\mathrm{tr}(q))\}$ is disjoint to $\mathscr{Y}$*
(b) *there is a function ζ such that $(\mathrm{Dom}(\zeta), \zeta) \in \mathrm{wfst}(\eta^*, \bar{D})$ and $\max(\mathrm{Dom}(\zeta)) \subseteq \mathscr{Y}$; that is*:
  (α) *$\mathrm{Dom}(\zeta)$ is a set Ξ satisfying*
    (i) $\Xi \subseteq \{\eta\colon \eta^* \trianglelefteq \eta \in {}^{\omega>}\omega\}$
    (ii) $\eta^* \in \Xi$
    (iii) *if $\eta \in \Xi$ and $\eta^* \trianglelefteq \nu \trianglelefteq \eta$ then $\nu \in \Xi$*
  (β) (i) $\mathrm{Rang}(\zeta) \subseteq \omega_1$
    (ii) $\eta^* \trianglelefteq \nu \triangleleft \eta \in \Xi \Rightarrow \zeta(\eta) < \zeta(\nu)$
  (γ) *for every $\eta \in \Xi$ at least one of the following holds*:
    (i) $\eta \in \mathscr{Y}$
    (ii) *the set $\{n\colon \eta^\frown\langle n\rangle \in \Xi\}$ belongs to D_η^+.*

Proof. Similar to [4, 4.7] or better [5, 5.4].

In full, recall $\Lambda = \{\eta\colon \eta^* \trianglelefteq \eta \in {}^{\omega>}\omega\}$. We define when $\mathrm{dp}(\eta) \geqslant \zeta$ for $\eta \in \Lambda$ by induction on the ordinal ζ:

$\boxplus$
- <u>$\zeta = 0$</u>: always
- <u>ζ a limit ordinal</u>: $\mathrm{dp}(\eta) \geqslant \zeta$ iff $\mathrm{rk}(\eta) \geqslant \xi$ for every $\xi < \zeta$
- <u>$\zeta = \xi + 1$</u>: $\mathrm{dp}(\eta) \geqslant \zeta$ <u>iff</u> both of the following occurs:
  (i) $\eta \notin \mathscr{Y}$
  (ii) the following set belongs to D_η^+: $\{n\colon \mathrm{dp}(\eta^\frown\langle n\rangle) \geqslant \xi\}$.

We define $\mathrm{dp}(\eta) \in \mathrm{Ord} \bigcup \{\infty\}$ such that $\xi = \mathrm{dp}(\eta)$ <u>iff</u> $(\forall \zeta \in \mathrm{Ord})[\mathrm{dp}(\eta) \geqslant \zeta \text{ iff } \zeta \leqslant \xi]$.

Easily

$\boxplus$ for every $\eta \in \Lambda, \mathrm{dp}(\eta) \in \omega_1 \cup \{\infty\}$.

Case 1: $\mathrm{dp}(\eta^*) = \infty$.

For each $\eta \in \Lambda$ such that $\mathrm{dp}(\eta) = \infty$ clearly there is $A_\eta \in D_\eta$ such that $n \in A_\eta \Rightarrow \mathrm{dp}(\eta^\frown\langle n\rangle) = \infty$. Let q be

$$\bigl\{\nu \in p\colon \nu \trianglelefteq \eta^* \text{ or } \eta^* \triangleleft \nu \text{ and if } \eta^* \trianglelefteq \rho \triangleleft \nu \text{ then } \nu\bigl(\ell g(\rho)\bigr) \in A_\rho\bigr\}.$$

Clearly q is as required in clause (a) of 1.8.

Case 2: $\mathrm{dp}(\eta^*) < \infty$.

We define

$$\Xi = \bigl\{\nu\colon \eta^* \trianglelefteq \nu \text{ and if } k \in \bigl[\ell g\bigl(\eta^*\bigr), \ell g(\nu)\bigr) \text{ then } \nu \restriction k \notin \mathscr{Y} \text{ and } \mathrm{dp}\bigl(\nu \restriction k\bigr) > \mathrm{dp}\bigl(\nu \restriction (k+1)\bigr)\bigr\}.$$

We define $\zeta : \Xi \to \omega_1$ by $\zeta(\eta) = \mathrm{dp}(\eta)$.

Now check. $\square_{1.8}$

Claim 1.9. $\mathbb{P}_1 * \mathbb{Q}_{\underset{\sim}{D}_1} \lessdot \mathbb{P}_2 * \mathbb{Q}_{\underset{\sim}{D}_2}$ <u>*when*</u>:

(a) *$\mathbb{P}_1 \lessdot \mathbb{P}_2$ and $\bar{\underset{\sim}{D}}_\ell = \langle \underset{\sim}{D}_{\ell,\eta}\colon \eta \in {}^{\omega>}\omega\rangle$ for $\ell = 1, 2$*
(b) *$\underset{\sim}{D}_{1,\eta}$ is a $\mathbb{P}_1$-name of a filter on $\mathbb{N}$*

(c) $\underset{\sim}{D}_{2,\eta}$ *is a* $\mathbb{P}_2$*-name of a filter on* $\mathbb{N}$

(d) $\Vdash_{\mathbb{P}_2}$ "$\underset{\sim}{D}_{1,\eta} \subseteq \underset{\sim}{D}_{2,\eta}$ *and moreover* $(\mathrm{fil}(\underset{\sim}{D}_{1,\eta})^+)^{\mathbf{V}[\mathbb{P}_1]} \subseteq \mathrm{fil}(\underset{\sim}{D}_{2,\eta})^+$, *i.e. for every* $A \in \mathscr{P}(\mathbb{N})^{\mathbf{V}[\mathbb{P}_1]}$ *we have* $A \in \mathrm{fil}(\underset{\sim}{D}_{1,\eta}) \Leftrightarrow A \in \mathrm{fil}(\underset{\sim}{D}_{2,\eta})$".

Proof. Like [4, §4] more [5, §5] but we elaborate.

Without loss of generality $\emptyset \in \mathbb{P}_1$ and $\emptyset \leqslant_{\mathbb{P}_2} p$ for every $p \in \mathbb{P}_2$. Clearly $\mathbb{P}_1 * \mathbb{Q}_{\underset{\sim}{\bar{D}}_1} \subseteq \mathbb{P}_2 * \mathbb{Q}_{\underset{\sim}{\bar{D}}_2}$ by clause (d) of the assumption and moreover $\mathbb{P}_1 \lessdot \mathbb{P}_2 \lessdot \mathbb{P}_2 * \mathbb{Q}_{\underset{\sim}{\bar{D}}_2}$ recalling Definition 1.1(1), (2). Now we can force by $\mathbb{P}_1$ so without loss of generality it is trivial, hence we have to prove that $\mathbb{Q}_{\bar{D}_1} \lessdot \mathbb{P}_2 * \mathbb{Q}_{\underset{\sim}{\bar{D}}_2}$ identifying $q \in \mathbb{Q}_{\bar{D}_1}$ with $(\emptyset, q) \in \mathbb{P}_2 * \mathbb{Q}_{\underset{\sim}{\bar{D}}_2}$. By clause (d) of the assumption, this identification is well defined and $\mathbb{Q}_{\bar{D}_1} \subseteq_{\mathrm{ic}} \mathbb{P}_2 * \mathbb{Q}_{\underset{\sim}{\bar{D}}_2}$ because for $p_1, p_2 \in \mathbb{Q}_{\bar{D}_1}$, p_1, p_2 are compatible iff $(\mathrm{tr}(p_1) \in p_2) \vee (\mathrm{tr}(p_2) \in p_1)$. It suffices to verify 1.1(3), requirement $\boxplus_2$. So let $(p_2, \underset{\sim}{q}_2) \in \mathbb{P}_2 * \mathbb{Q}_{\underset{\sim}{D}_2}$; without loss of generality for some η^* from $\mathbf{V}$ we have $p_2 \Vdash$ "$\eta^* = \mathrm{tr}(\underset{\sim}{q}_2)$", so $\eta^* \in {}^{\omega>}\omega$ and of course:

$(*)_1$ $\Vdash_{\mathbb{P}_2}$ "$\underset{\sim}{q}_2 \in \mathbb{Q}_{\underset{\sim}{\bar{D}}_2}$".

By 1.1(3), it suffices to find $q \in \mathbb{Q}_{\bar{D}_1}$ such that

$(*)_2$ $q \leqslant q' \in \mathbb{Q}_{\bar{D}_1} \Rightarrow (p_2, \underset{\sim}{q}_2), q'$ are compatible; that is, $(p_2, \underset{\sim}{q}_2), (\emptyset, q')$ are compatible in $\mathbb{P}_2 * \mathbb{Q}_{\underset{\sim}{\bar{D}}_2}$.

Now we shall apply Claim 1.8 in $\mathbf{V}$ with $\eta^*, \bar{D}_1$ here standing for $\eta^*, \bar{D}$ there. Still $\mathscr{Y}$ is missing, so let

$$\begin{aligned}\mathscr{Y} = \big\{\nu\colon\ & \eta^* \trianglelefteq \nu \in {}^{\omega>}\omega \\ & \text{and there is } r \in \mathbb{Q}_{\bar{D}_1} \text{ such that } \nu = \mathrm{tr}(r) \text{ and} \\ & (\emptyset, r), (p_2, \underset{\sim}{q}_2) \text{ are incompatible in } \mathbb{P}_2 * \mathbb{Q}_{\underset{\sim}{\bar{D}}_2} \\ & \text{equivalently } p_2 \Vdash_{\mathbb{P}_2} \text{“}\underset{\sim}{q}_2, r \text{ are incompatible in } \mathbb{Q}_{\underset{\sim}{\bar{D}}_2}\text{”}\big\}.\end{aligned}$$

By Claim 1.8 below we get clause (a) or clause (b) there.

Case 1: Clause (a) holds, say as witnessed by $q \in \mathbb{Q}_{\bar{D}_1}$.

We shall prove that in this case q is as required, i.e. $q \in \mathbb{Q}_{\bar{D}_1}$ and $[q \leqslant_{\mathbb{Q}_{\bar{D}_1}} r \in \mathbb{Q}_{\bar{D}_1} \Rightarrow (p_2, \underset{\sim}{q}_2) \in \mathbb{P}_2 * \mathbb{Q}_{\underset{\sim}{\bar{D}}_2}$ and r are compatible (in $\mathbb{P}_2 * \mathbb{Q}_{\underset{\sim}{\bar{D}}_2}$)].

Why? Let $\nu = \mathrm{tr}(r)$. Clearly $(\eta^* \trianglelefteq \nu \in q)$ hence by the choice of q, i.e. 1.8(a)(β) we have $\nu \notin \mathscr{Y}$ so r cannot witness "$\nu \in \mathscr{Y}$" hence $r, (p_2, \underset{\sim}{q}_2)$ are compatible in $\mathbb{P}_2 * \mathbb{Q}_{\underset{\sim}{\bar{D}}_2}$ as required.

Case 2: Clause (b) holds as witnessed by the function ζ.

By the definition of $\mathscr{Y}$, in $\mathbf{V}$, we can choose $\bar{q}$ such that:

$\boxplus$ (a) $\bar{q} = \langle q_\nu\colon\ \nu \in \mathscr{Y}\rangle$

(b) $q_\nu \in \mathbb{Q}_{\bar{D}_1}$ and $\mathrm{tr}(q_\nu) = \nu$

(c) q_ν witness $\nu \in \mathscr{Y}$, i.e. $p_2 \Vdash$ "$q_\nu, \underset{\sim}{q}_2$ are incompatible in $\mathbb{Q}_{\underset{\sim}{\bar{D}}_2}$".

We define a $\mathbb{P}_2$-name $\underset{\sim}{q}_*$ as follows:

$$\begin{aligned}\underset{\sim}{q}_* = \big\{\nu\colon\ & \nu \trianglelefteq \eta^* \ \underline{\text{or}}\ \eta^* \triangleleft \nu \in \underset{\sim}{q}_2 \text{ and if } \ell g(\eta^*) \leqslant k < \ell g(\nu) \\ & \text{and } \nu{\restriction}k \in \mathscr{Y} \text{ then } \nu \in q_{\nu\restriction k}, \text{ hence} \\ & k \leqslant \ell \leqslant \ell g(\nu) \Rightarrow \nu{\restriction}\ell \in q_{\nu\restriction k}\big\}.\end{aligned}$$

Clearly $\Vdash_{\mathbb{P}_2}$ "$\underset{\sim}{q}_* \in \mathbb{Q}_{\underset{\sim}{\bar{D}}_2}$ and $\mathrm{tr}(\underset{\sim}{q}_*) = \eta^*$ and $\mathbb{Q}_{\underset{\sim}{\bar{D}}_2} \models$ "$\underset{\sim}{q}_2 \leqslant \underset{\sim}{q}_*$"".

$(*)_3$ if $\nu \in \mathscr{Y}$ then $\eta^* \trianglelefteq \nu$ and $p_2 \Vdash_{\mathbb{P}_2}$ "$\neg(\nu \in \underset{\sim}{q}_*)$".

[Why? Otherwise there is $p_3 \in \mathbb{P}_2$ such that $p_2 \leqslant p_3$ and $p_3 \Vdash_{\mathbb{P}_2}$ "$\eta^* \trianglelefteq \nu \in \underset{\sim}{q}_*$", as $\mathrm{tr}(\underset{\sim}{q}_*)$ is forced to be η^* and $\mathrm{tr}(q_\nu) = \nu$, necessarily $p_3 \Vdash_{\mathbb{P}_2}$ "$q_\nu, \underset{\sim}{q}_*$ are compatible". But $p_2 \Vdash_{\mathbb{P}_2}$ "$\underset{\sim}{q}_2 \leqslant \underset{\sim}{q}_*$", we get a contradiction to the choice of q_ν.]

Now we know that $\eta_* \in \mathrm{Dom}(\zeta)$ and $\Vdash$ "$\eta^* \in \underset{\sim}{q}_*$" hence $\underset{\sim}{S} := \{\nu\colon\ \nu \in \mathrm{Dom}(\zeta)$ hence $\eta^* \trianglelefteq \nu$ and $p_2 \nVdash$ "$\nu \notin \underset{\sim}{q}_*$"$\}$ is not empty. So as $S \subseteq \mathrm{Dom}(\zeta)$ the set $\mathscr{U} = \{\zeta(\nu)\colon\ \nu \in \underset{\sim}{S}\}$ is not empty, and by the choice of the function ζ we have $\underset{\sim}{\mathscr{U}} \subseteq \omega_1$, hence there is a minimal $\gamma \in \mathscr{U}$ and let $\nu \in \mathrm{Dom}(\zeta)$ be such that $\zeta(\nu) = \gamma$. By the definition, if $\gamma = 0$ then by clauses (γ) and (β) of 1.8(b), i.e. the choice of $\zeta(-)$ we have $\nu \in \mathscr{Y}$ and, of course, $\nu \in S$. By $(*)_3$, $p_2 \Vdash_{\mathbb{P}_2}$ "$\neg(\nu \in \underset{\sim}{q}_*)$" we get easy contradiction to $\nu \in S$, hence we can assume $\gamma > 0$. By the definition of S there is $p_* \in \mathbb{P}_2$ such that $\mathbb{P}_2 \models$ "$p_2 \leqslant p_*$" and

$p_* \Vdash_{\mathbb{P}_2}$ "$\nu \in \underset{\sim}{q}_*$ hence $\in \underset{\sim}{q}_2$" and, of course, $\nu \in S$. By the choice of the function ζ, in **V** we have $A := \{n\colon \nu\hat{}\langle n\rangle \in \mathrm{Dom}(\zeta)\} \neq \emptyset$ mod $D_{1,\nu}$, hence by clause (d) of the assumption of the claim $\Vdash_{\mathbb{P}_2}$ "$A \neq \emptyset$ mod $\underset{\sim}{D}_{2,\nu}$" and, of course, $p_* \Vdash_{\mathbb{P}_2}$ "$\{n\colon \nu\hat{}\langle n\rangle \in \underset{\sim}{q}_*\} \in \underset{\sim}{D}_{2,\nu}$". Together $p_* \Vdash_{\mathbb{P}_2}$ "there is n such that $\nu\hat{}\langle n\rangle \in \underset{\sim}{q}_* \cap \mathrm{Dom}(\zeta)$", so let n_* and $p_{**} \in \mathbb{P}_2$ be such that $\mathbb{P}_2 \models$ "$p_* \leqslant p_{**}$" and $p_{**} \Vdash_{\mathbb{P}_2}$ "$\nu\hat{}\langle n_*\rangle \in \underset{\sim}{q}_* \cap \mathrm{Dom}(\zeta)$".

So $\zeta(\nu\hat{}\langle n_*\rangle)$ is well defined, i.e. $\nu\hat{}\langle n_*\rangle$ belongs to $\mathrm{Dom}(\zeta)$ hence $\zeta(\nu\hat{}\langle n_*\rangle) < \zeta(\nu) = \gamma$ and easily $\nu\hat{}\langle n_*\rangle \in S$ and $\zeta(\nu\hat{}\langle n_*\rangle) \in \mathscr{U}$, so we get a contradiction to the choice of γ. $\square_{1.9}$

Definition 1.10. 1) We say $\mathbf{d} = (\bar{D}, F)$ is a frame <u>when</u>:

(a) $\bar{D} = \langle D_\eta\colon \eta \in {}^{\omega>}\omega\rangle$ and $D_\eta \subseteq [\mathbb{N}]^{\aleph_0}$, $\emptyset \notin \mathrm{fil}(D_\eta)$ for $\eta \in {}^{\omega>}\omega$
(b) $F \subseteq [\mathbb{N}]^{\aleph_0}$ and $\emptyset \notin \mathrm{fil}(F)$.

1A) Above let $\bar{D}_{\mathbf{d}} = \langle D_{\mathbf{d},\eta}\colon \eta \in {}^{\omega>}\omega\rangle$, $D_{\mathbf{d},\eta} = \mathrm{fil}(D_\eta)$, $F_{\mathbf{d}} = \mathrm{fil}(F)$, $\mathbb{Q}_{\mathbf{d}} = \mathbb{Q}_{\bar{D}_{\mathbf{d}}}$ and if $D_\eta = D$ for $\eta \in {}^{\omega>}\omega$ we may write $D, D_{\mathbf{d}}$ instead of $\bar{D}, \bar{D}_{\mathbf{d}}$, respectively.

2) We say $\underset{\sim}{A}$ is a $\mathbf{d}$-candidate <u>when</u> ($\mathbf{d}$ is a frame and):

(c) $\underset{\sim}{A}$ is a $\mathbb{Q}_{\mathbf{d}}$-name of a subset of $\mathbb{N}$.

3) We say $\underset{\sim}{A}$ is $\mathbf{d}$-null when it is a $\mathbf{d}$-candidate and is not $\mathbf{d}$-positive, see below.

4) We say $\underset{\sim}{A}$ is $\mathbf{d}$-positive <u>when</u> for some $p_* \in \mathbb{Q}_{\mathbf{d}}$, for a dense set of $p \geqslant p_*$ some quadruple $(p, A, \bar{S}, \bar{\zeta})$ is a local witness[2] for $(\underset{\sim}{A}, \mathbf{d})$ <u>or</u> for $(\eta, \underset{\sim}{A}, \mathbf{d})$ when $\eta = \mathrm{tr}(p)$ <u>or</u> for $(p, \underset{\sim}{A}, \mathbf{d})$ <u>or</u> for $\underset{\sim}{A}$ being $\mathbf{d}$-positive, which means:

(a) $p \in \mathbb{Q}_{\mathbf{d}}$
(b) $A \in F^+_{\mathbf{d}}$
(c) $\bar{S} = \langle S_n\colon n \in A\rangle$ and $\bar{\zeta} = \langle \zeta_n\colon n \in A\rangle$
(d) $(S_n, \zeta_n) \in \mathrm{wfst}(p, \bar{D})$ for $n \in A$ recalling Definition 1.7(2)
(e) if $\eta \in S_n$ and $\zeta_n(\eta) = 0$ then $p^{[\eta]} \Vdash$ "$n \in \underset{\sim}{A}$".

Definition 1.11. 1) For a frame $\mathbf{d} = (\bar{D}, F)$ let $\underset{\sim}{\mathrm{id}}_{\mathbf{d}} = \underset{\sim}{\mathrm{id}}(\mathbf{d}) = \{\underset{\sim}{A} \subseteq \mathbb{N}\colon \underset{\sim}{A}$ is a $\mathbb{Q}_{\mathbf{d}}$-name which is $\mathbf{d}$-null$\}$.

2) If $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbf{d}}$ is a frame" then $\underset{\sim}{\mathrm{id}}_{\underset{\sim}{\mathbf{d}}}[\mathbb{P}]$ is the $\mathbb{P} * \mathbb{Q}_{\underset{\sim}{\mathbf{d}}}$-name of $\underset{\sim}{\mathrm{id}}_{\underset{\sim}{\mathbf{d}}}$.

Claim 1.12. *For a frame $\mathbf{d}$, $\Vdash_{\mathbb{Q}_{\bar{D}}}$ "$\underset{\sim}{\mathrm{id}}_{\mathbf{d}}$ is an ideal on $\mathbb{N}$ containing the finite sets and $\mathbb{N} \notin \underset{\sim}{\mathrm{id}}_{\mathbf{d}}$"; moreover, for every $A \in \mathscr{P}(\mathbb{N})$ from* **V**, *we have $A = \emptyset$ mod $F_{\mathbf{d}}$ iff $\Vdash_{\mathbb{Q}_{\mathbf{d}}}$ "$A \in \underset{\sim}{\mathrm{id}}_{\mathbf{d}}$".*

Proof. It suffices to prove the following $\boxplus_1 - \boxplus_4$.

$\boxplus_1$ If $\Vdash_{\mathbb{Q}_{\bar{D}}}$ "if $A_1 \subseteq A_2$ and $A_2 \in \underset{\sim}{\mathrm{id}}_{\mathbf{d}}$ then $A_1 \in \underset{\sim}{\mathrm{id}}_{\mathbf{d}}$".

[Why? If $(p, A, \bar{S}, \bar{\zeta})$ is a local witness for $(\underset{\sim}{A}_1, \mathbf{d})$ <u>then</u> obviously it is a local witness for $(\underset{\sim}{A}_2, \mathbf{d})$.]

$\boxplus_2$ if $\Vdash_{\mathbb{Q}_{\mathbf{d}}}$ "if $A_1, A_2 \in \underset{\sim}{\mathrm{id}}_{\mathbf{d}}$ then $A_1 \cup A_2 \in \underset{\sim}{\mathrm{id}}_{\mathbf{d}}$".

Why? It suffices to prove: if $\Vdash_{\mathbb{Q}_{\mathbf{d}}}$ "$\underset{\sim}{A}_1 \cup \underset{\sim}{A}_2 = \underset{\sim}{A} \subseteq \mathbb{N}$" and $\underset{\sim}{A}$ is $\mathbf{d}$-positive <u>then</u> $\underset{\sim}{A}_\ell$ is $\mathbf{d}$-positive for some $\ell \in \{1,2\}$. Let $(p, A, \bar{S}, \bar{\zeta})$ be a local witness for $(\underset{\sim}{A}, \mathbf{d})$ and we shall prove that there are $\ell \in \{1,2\}$ and a local witness for $(\mathrm{tr}(p), \underset{\sim}{A}_\ell, \mathbf{d})$; by the "dense" in Definition 1.10(4) this suffices.

For any $n \in A$ and $\nu \in S_n$ such that $\zeta_n(\nu) = 0$ we choose $(\ell_{n,\nu}, \zeta_{n,\nu}, S_{n,\nu})$ such that:

$(*)_{2.1}$ (a) $\ell_{\nu,n} \in \{1,2\}$
(b) $(S_{n,\nu}, \zeta_{n,\nu}) \in \mathrm{wfst}(p^{[\nu]}, \bar{D}_{\mathbf{d}})$
(c) if $\zeta_{n,\nu}(\rho) = 0$ so $\rho \in S_{n,\nu}$ <u>then</u> there is $q \in \mathbb{Q}_{\mathbf{d}}$ such that $p \leqslant q$, $\mathrm{tr}(q) = \rho$ and $q \Vdash$ "$n \in \underset{\sim}{A}_{\ell_{n,\nu}}$"; let $q_{n,\rho}$ be such q.

[Why $(\rho_{n,\nu}, \zeta_{n,\nu}, S_{n,\nu})$ exists? We shall use 1.8; that is for $\ell \in \{1,2\}$ let $\mathscr{Y}_{n,\nu,\ell} = \{\rho\colon \nu \trianglelefteq \rho \in p$ and there is $r \in \mathbb{Q}_{\bar{D}}$ such that $\mathrm{tr}(r) = \rho$ and $p \leqslant r$ and $r \Vdash$ "$n \in \underset{\sim}{A}_\ell$"$\}$.

We apply for $\ell = 1, 2$ Claim 1.8 with $\bar{D}_{\mathbf{d}}, \nu, \mathscr{Y}_{n,\nu,\ell}$ here standing for $\bar{D}, \eta^*, \mathscr{Y}$ there. If for some $\ell \in \{1,2\}$ clause (b) there holds as witness by the function ζ, easily the desired $(*)_{2.1}$ holds. If for both $\ell = 1, 2$ clause (a) there holds then for $\ell = 1, 2$ there is $q_\ell \in \mathbb{Q}_{\mathbf{d}}$ such that $\mathrm{tr}(q_\ell) = \nu$ and $q_\ell \cap \mathscr{Y}_{n,\nu,\ell} = \emptyset$.

[2] An equivalent version is when we weaken clause (e) to: if $\eta \in S_n$ and $\zeta_n(\eta) = 0$ then there is $q \in \mathbb{Q}_{\mathbf{d}}$ such that $\mathrm{tr}(q) = \eta$, $p \leqslant q$ and $q \Vdash$ "$n \in \underset{\sim}{A}$", see $(*)_{2.2}$ in the proof. Moreover, we can omit "$p \leqslant q$"; hence actually only $\mathrm{tr}(p)$ is important so we may write $\mathrm{tr}(p)$ instead of p.

Necessarily $q := q_1 \cap q_2 \cap p$ belongs to $\mathbb{Q}_{\mathbf{d}}$ and has trunk ν and is disjoint to $\mathscr{Y}_{n,\nu,1} \cup \mathscr{Y}_{n,\nu,2}$. But $\mathbb{Q}_{\mathbf{d}} \models$ "$p^{[\nu]} \leqslant q$" and $q^{[\nu]} \Vdash_{\mathbb{Q}_{\mathbf{d}}}$ "$n \in \underset{\sim}{A} = \underset{\sim}{A}_1 \cup \underset{\sim}{A}_2$", hence there are $\ell \in \{1,2\}$ and $r \in \mathbb{Q}_{\mathbf{d}}$ such that $q \leqslant r$ and $r \Vdash_{\mathbb{Q}_{\mathbf{d}}}$ "$n \in \underset{\sim}{A}_\ell$", but then $\mathrm{tr}(r) \in \mathscr{Y}_{n,\nu,\ell}$ and $\mathrm{tr}(r) \in q_* \subseteq q_\ell$, contradicting the choice of q_ℓ. So $(*)_{2.1}$ holds indeed.]

$(*)_{2.2}$ without loss of generality $\zeta_{n,\nu}(\rho) = 0 \Rightarrow \ell g(\rho) > n$.

[Why? Obvious.]

$(*)_{2.3}$ for $n \in A$ there are ℓ_n, S'_n, ζ'_n such that
- (a) $(S'_n, \zeta'_n) \in \mathrm{wfst}(p, \bar{D}_{\mathbf{d}})$
- (b) $S'_n \subseteq S_n$ and $\max(S'_n) = S'_n \cap \max(S_n)$
- (c) $\ell_n \in \{1,2\}$ and $\nu \in \max(S'_n) \Rightarrow \ell_{n,\nu} = \ell_n$.

[Why? Easy.]

$(*)_{2.4}$ for $n \in A$ letting $S''_n = \bigcup\{S_{n,\nu}\colon\ \nu \in \max(S'_n)\} \cup S'_n$, for some ζ''_n and $\bar{q}_n$ we have:
- $(S''_n, \zeta''_n) \in \mathrm{wfst}(p, \bar{D})$
- $\{\rho\colon\ \zeta''_n(\rho) = 0\} = \{\rho\colon$ for some ν we have $\nu \in S'_n,\ \zeta_n(\nu) = 0,\ \rho \in S_{n,\nu}$ and $\zeta_{n,\nu}(\rho) = 0\}$
- $\bar{q} = \langle q_{n,\rho}\colon\ \zeta''_n(\rho) = 0\rangle$
- $\zeta''_n(\rho) = 0 \Rightarrow p \leqslant q_{n,\rho}$
- $\mathrm{tr}(q_{n,\rho}) = \rho$
- $q_{n,\rho} \Vdash$ "$n \in \underset{\sim}{A}_{\ell_n}$".

[Why? Think.]

$(*)_{2.5}$ there is $\ell \in \{1,2\}$ such that $A' := \{n \in A\colon\ \ell_n = \ell\} \neq \emptyset \bmod F_{\mathbf{d}}$.

[Why? Obvious as $A \in F^+_{\mathbf{d}}$.]

We now consider the quadruple $(p', A', \bar{S}'', \bar{\zeta}'')$ defined by:

- $p' = \{\varrho \in p$: if $\mathrm{tr}(p) \trianglelefteq \rho \trianglelefteq \varrho, n \leqslant \ell g(\rho)$ and $\rho \in \max(S''_n)$ then $\varrho \in q_{n,\rho}\}$ where $S''_n, q_{n,\rho}$ are from $(*)_{2.4}$.

[Why $p' \in \mathbb{Q}_{\mathbf{d}}$ with $\mathrm{tr}(p') = \mathrm{tr}(p)$? Recall $(*)_{2.2}$.]

So together we have:

- A' is from $(*)_{2.5}$, so $A' \in F^+_{\mathbf{d}}$
- $\bar{S}'' = \langle S''_n\colon\ n \in A'\rangle$ where S''_n is from $(*)_{2.4}$
- $\bar{\zeta}'' = \langle \zeta''_n\colon\ n \in A'\rangle$ where ζ''_n is from $(*)_{2.4}$.

Now check that $(p', A', \bar{S}'', \bar{\zeta}'')$ is a local witness for $(\mathrm{tr}(p), \underset{\sim}{A}_\ell, \bar{D})$ hence $\boxplus_2$ holds as said in the beginning of its proof.

$\boxplus_3$ $\Vdash_{\mathbb{Q}_{\mathbf{d}}}$ "$\emptyset \in \underset{\sim}{\mathrm{id}}_{\mathbf{d}}$; moreover if $A = \emptyset \bmod F_{\mathbf{d}}$ is from $\mathbf{V}$ then $A \in \underset{\sim}{\mathrm{id}}_{\mathbf{d}}$".

Why? Because of clause (b) in Definition 1.10(4).

$\boxplus_4$ $\Vdash_{\mathbb{Q}_{\bar{D}}[\mathbf{d}]}$ "$\mathbb{N} \notin \underset{\sim}{\mathrm{id}}_{\mathbf{d}}$, moreover if $B \in F^+_{\mathbf{d}}$ and $B \in \mathbf{V}$ then $B \notin \underset{\sim}{\mathrm{id}}_{\mathbf{d}}$".

Why? This means that B is $\mathbf{d}$-positive which is obvious: use the local witness $(p, A, \bar{S}, \bar{\zeta})$ where p is any member of $\mathbb{Q}_{\mathbf{d}}$, $A = B$, $S_n = \{\mathrm{tr}(p)\}$, $\zeta_n(\mathrm{tr}(p)) = 0$. $\square_{1.12}$

Observation 1.13. Assume $\mathbf{d}_1, \mathbf{d}_2$ are frames and $\bar{D}_{\mathbf{d}_1} = \bar{D} = \bar{D}_{\mathbf{d}_2}$ and $F_{\mathbf{d}_1} \subseteq F_{\mathbf{d}_2}$ <u>then</u> $\Vdash_{\mathbb{Q}_{\bar{D}}}$ "$\underset{\sim}{\mathrm{id}}_{\mathbf{d}_1} \subseteq \underset{\sim}{\mathrm{id}}_{\mathbf{d}_2}$".

Proof. Should be clear. $\square_{1.13}$

Claim 1.14. *We have* $\Vdash_{\mathbb{P}_2}$ "$\underset{\sim}{\mathrm{id}}_{\underset{\sim}{\mathbf{d}}_1} \subseteq \underset{\sim}{\mathrm{id}}_{\underset{\sim}{\mathbf{d}}_2}$ *and* $(\underset{\sim}{\mathrm{id}}_{\underset{\sim}{\mathbf{d}}_1})^+[\mathbb{P}_1] \subseteq (\underset{\sim}{\mathrm{id}}_{\underset{\sim}{\mathbf{d}}_2})^+[\mathbb{P}_2]$" <u>*when*</u>:

(a) $\mathbb{P}_1 \lessdot \mathbb{P}_2$
(b) $\Vdash_{\mathbb{P}_\ell}$ "$\underset{\sim}{\mathbf{d}}_\ell$ *is a frame*" *for* $\ell = 1, 2$
(c) $\Vdash_{\mathbb{P}_2}$ "$\underset{\sim}{D}_{\mathbf{d}_1,\eta} \subseteq \underset{\sim}{D}_{\mathbf{d}_2,\eta}$" *for* $\eta \in {}^{\omega>}\omega$

(d) *if* $A \in (D^+_{\underset{\sim}{\mathbf{d}}_1,\eta})^{\mathbf{V}[\mathbb{P}_1]}$ *then* $A \in (D^+_{\underset{\sim}{\mathbf{d}}_2})^{\mathbf{V}[\mathbb{P}_2]}$
(e) $\Vdash_{\mathbb{P}_2}$ "$F_{\underset{\sim}{\mathbf{d}}_1} \subseteq F_{\underset{\sim}{\mathbf{d}}_2}$"
(f) *if* $A \in (F^+_{\underset{\sim}{\mathbf{d}}_1})^{\mathbf{V}[\mathbb{P}_1]}$ *then* $A \in (F^+_{\underset{\sim}{\mathbf{d}}_2})^{\mathbf{V}[\mathbb{P}_2]}$.

Proof. Should be clear by 1.15 below recalling 1.9. $\square_{1.14}$

Claim 1.15. *Let* $\mathbf{d}$ *be a frame and* $\underset{\sim}{A}$ *a* $\mathbb{Q}_{\bar{D}_\mathbf{d}}$*-name of a subset of* $\mathbb{N}$*. We have* $\underset{\sim}{A}$ *is* $\mathbf{d}$*-null* <u>*iff*</u> *for a pre-dense set of* $p \in \mathbb{Q}_\mathbf{d}$ *we have* $\mathrm{tr}(p) \trianglelefteq \rho \in p \Rightarrow$ *there is no local witness for* $(p^{[\rho]}, \underset{\sim}{A}, \mathbf{d})$ *equivalently, for* $(\rho, \underset{\sim}{A}, \mathbf{d})$.

Proof. Straight. $\square_{1.15}$

Remark 1.16. The point of 1.15 is that the second condition is clearly absolute in the relevant cases by 1.9, i.e. in 1.14.

Definition 1.17. 1) $\mathrm{fin}(I)$ is the set of finite functions from I to $\mathscr{H}(\aleph_0)$.

2) Let $\mathbf{K}$ be the set of forcing notions $\mathbb{Q}$ such that some pair (I, f) witness it, i.e. $(I, f, \mathbb{Q}) \in \mathbf{K}^+$ which means:

(a) f is a function from $\mathbb{Q}$ to $\mathrm{fin}(I)$
(b) if $p_1, p_2 \in \mathbb{Q}$ and the functions $g(p_1), g(p_2)$ are compatible <u>then</u> p_1, p_2 have a common upper bound p with $g(p) = g(p_1) \cup g(p_2)$.

2) We define $\leqslant_\mathbf{K} = \leqslant^{\mathrm{wk}}_\mathbf{K}$ by: $(I_1, f_1, \mathbb{Q}_1) \leqslant^{\mathrm{wk}}_\mathbf{K} (I_2, f_2, \mathbb{Q}_2)$ means that:

(a) (I_ℓ, f_ℓ) witness $\mathbb{Q}_\ell \in \mathbf{K}$ for $\ell = 1, 2$
(b) $I_1 \subseteq I_2$
(c) $f_1 \subseteq f_2$
(d) $\mathbb{Q}_1 \subseteq_{\mathrm{ic}} \mathbb{Q}_2$.

3) We define $\leqslant^{\mathrm{st}}_\mathbf{K}$ similarly adding:

(d)$^+$ $\mathbb{Q}_1 \lessdot \mathbb{Q}_2$.

4) If $\mathbf{q} \in \mathbf{K}^+$ let $\mathbf{q} = (I_\mathbf{q}, f_\mathbf{q}, \mathbb{Q}_\mathbf{q})$.

Remark 1.18. We can use much less in Definition 1.17.

2. Consistency of many gaps

We prove the first result promised in the introduction. Assume $\lambda = \lambda^{<\lambda} > \aleph_1$ and we like to build a c.c.c. forcing notion $\mathbb{P}$ of cardinality λ, such that $\mathbf{V}^\mathbb{P}$ is as required: Sp_χ includes Θ_1 and is disjoint to Θ_2; really we force by $\mathbb{P} \times \prod_\theta \mathscr{T}_\theta$, the $\mathscr{T}_\theta$ quite complete and translate $\mathbb{P}$-names of ultra systems of filters to ultra-filters. In order to have $\Theta_1 \subseteq \mathrm{Sp}_\chi$, we shall represent $\mathbb{P}$ as an FS iteration $\langle P_\alpha, \mathbb{Q}_\beta \colon \alpha \leqslant \delta,\ \beta < \delta \rangle$, $|\mathbb{P}_\alpha| \leqslant \lambda$ and $\mathscr{T}_\theta$ is, e.g. ${}^{\theta>}2$ and for each $\theta \in \Theta_2$ we have a $\bar{D}_\alpha = \langle \underset{\sim}{D}_{\alpha,s} \colon s \in \mathscr{T}_\theta \rangle$ a $\mathbb{P}_\alpha$-name of an ultra system of filters for unboundedly many $\alpha < \delta$, increasing with α; in the end we force by $\mathbb{P}_\alpha \times \Pi\{\mathscr{T}_\theta \colon \theta \in \Theta_1\}$. Toward this for each $s \in \mathscr{T}_\theta, \theta \in \Theta_1$ we many times force by $\mathbb{Q}^+_{D_{\alpha,s}}$ from Section 1.

But in order to have $\Theta_2 \cap \mathrm{Sp}_\chi = \emptyset$, we intend to represent $\mathbb{P}$ as the union of a $\lessdot$-increasing sequence $\langle \mathbb{P}'_\varepsilon \colon \varepsilon < \lambda \rangle$ and for each $\theta \in \Theta_2$ for stationarily many $\varepsilon < \lambda, \mathrm{cf}(\varepsilon) = \theta$ and $\mathbb{P}'_{\varepsilon+1}$ is essentially the ultrapower $(\mathbb{P}'_\varepsilon)^\theta / E_\theta, E_\theta$ a θ-complete ultra-filter on θ, so θ is a measurable cardinal.

To accomplish both we define a set $\mathbf{Q}$, each $\mathbf{x} \in \mathbf{Q}$ consist of an FS iteration of $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta \colon \alpha \leqslant \lambda^+,\ \beta < \lambda^+ \rangle$ with $\langle \bar{\underset{\sim}{D}}_{s,\alpha} \colon s \in \bigcup\{\mathscr{T}_\theta \colon \theta \in \Theta_1\} \rangle$ for many $\alpha < \lambda^+$, increasing with α and $\mathbb{Q}_\beta = \mathbb{Q}_{D_{t(\beta),\alpha}}$.

In the end for suitable $\mathbf{x}$, we shall use $\mathbb{P}_\delta$ for some $\delta < \lambda^+$ of cofinality $\kappa \ll \lambda$ (e.g. $\kappa = \aleph_1$). So why go so high as λ^+? It helps in the construction toward the other aim; we shall construct $\langle \mathbf{x}_\varepsilon \colon \varepsilon \leqslant \lambda \rangle$ increasing in $\mathbf{Q}$ such that for each $\theta \in \Theta_2$ for $\varepsilon < \lambda$ of cofinality $\theta, \mathbf{x}_{\varepsilon+1}$ is essentially $(\mathbf{x}_\varepsilon)^\theta / E_\theta$. In particular, we have to prove $\mathbf{Q} \neq \emptyset$, the existence of the ultrapower and the existence of limit which happens to be a major proof here. For this we have to choose the right definition, in particular using $\underset{\sim}{\mathrm{id}}_{(D_{\alpha,s}, D_{\beta,t})}$ from Definition 1.11.

For this section we assume:

Hypothesis 2.1. 1) We now fix two cardinals κ and λ as well as two sets, Θ_1 and Θ_2, of regular cardinals in the interval $[\kappa, \lambda]$ and let $\Theta = \Theta_1 \cup \Theta_2$.

Our assumptions are:

(a) κ is regular and uncountable, $\lambda = \lambda^{\aleph_0}$ and $\kappa < \lambda$
(b) Θ_1 and Θ_2 are disjoint sets of regular cardinals $< \lambda$ from the interval $[\kappa, \lambda)$ but $\kappa \notin \Theta_2$
(c) each $\theta \in \Theta_1$ we have $\theta^{<\theta} = \theta$
(d) each $\theta \in \Theta_2$ carries a normal ultrafilter E_θ, hence Θ_2 consists of measurable cardinals
(e) for all $\theta \in \Theta_2$ the cardinal λ satisfies $\mathrm{cf}(\lambda) > \theta$ and $\lambda = \lambda^\theta / E_\theta$.

2) Furthermore (and see 2.4 below so it is not a burden)

(f) $\bar{\mathscr{T}} = \langle \mathscr{T}_\theta\colon \theta \in \Theta_1 \rangle$, $\mathscr{T}_\theta$ is a tree of cardinality θ with θ levels, such that above any element there are elements of any higher level (may add "$\mathscr{T}_\theta$ is $\aleph_2$-complete" and even "$\mathscr{T}_\theta$ is θ-complete", then clause (g) follows)
(g) for every $\partial \in \Theta_1$, forcing by $\mathscr{T}_{\geqslant \partial} := \Pi\{\mathscr{T}_\theta\colon \theta \in \Theta_1 \backslash \partial\}$, the product with Easton support, adds no sequence of ordinals of length $< \partial$ and, for simplicity, collapses no cardinal and changes no cofinality; if $\kappa = \aleph_1 \in \Theta$ add "$\mathscr{T}_\kappa$ is $\aleph_1$-complete"; let $\mathscr{T}_* = \mathscr{T}_{\geqslant \min(\Theta_1)}$
(h) if $\partial \in \Theta_1$ then $|\mathscr{T}_{\geqslant \partial}|$ is $\Pi(\Theta_1 \backslash \partial)$ except when $\sup(\Theta_1)$ is strongly inaccessible and then the value is $\sup(\Theta_1)$.

Choice 2.2. 1) Without loss of generality $\langle \mathscr{T}_\theta\colon \theta \in \Theta_1 \rangle$ is a sequence of pairwise disjoint trees.

2) Let $\mathscr{T}$ be the disjoint sum of $\{\mathscr{T}_\theta\colon \theta \in \Theta\}$, so it is a forest.

3) Let $\bar{t} = \langle t_i\colon i \in S \rangle$ be a sequence of members of $\mathscr{T}$ where $S = \{\delta < \lambda^+\colon \mathrm{cf}(\delta) = \mathrm{cf}(\lambda)\}$ such that if $t \in \mathscr{T}$ then $\{\delta \in S\colon t_\delta = t\}$ is a stationary subset of λ^+; let $t(i) = t_i$.

4) Furthermore choose

(α) $S_0 = \{\delta < \lambda^+\colon \mathrm{cf}(\delta) = \aleph_0\}$ is stationary
(β) $\bar{\Upsilon} = \langle \Upsilon_{\delta,t,n}\colon \delta \in S_0,\ t \in \mathscr{T},\ n \in \mathbb{N} \rangle$; let $\Upsilon(\delta, t, n) = \Upsilon_{\delta,t,n}$
(γ) $\langle \Upsilon_{\delta,t,n}\colon n < \omega \rangle$ is an increasing ω-sequence of ordinals with limit δ
(δ) $\Upsilon_{\delta,t,n} \in \{\alpha \in S\colon t_\alpha = t\}$
(ε) $\bar{\Upsilon}$ guess clubs, i.e. if E is a club of λ^+ then the set $\{\delta \in S_0\colon C^*_\delta := \{\Upsilon_{\delta,t,n}\colon t, n\} \subseteq E\}$ is stationary.

Remark 2.3. If $|\mathscr{T}| < \lambda$ we can find such $\bar{\Upsilon}$, but in general it is easy to force such $\bar{\Upsilon}$.

Claim 2.4. *Assuming* 2.1(1) *only, a sequence* $\bar{\mathscr{T}}$ *as in* 2.1, *clauses* (f), (g), (h) (*and also* $\bar{t}, s, S_\theta, \bar{\Upsilon}$ *as in* 2.2) *exists, provided that* $\Theta_1 \subseteq \{\theta\colon \theta = \theta^{<\theta} \geqslant \kappa\}$ *and GCH holds* (*or just* $\theta = \sup(\Theta_1 \cap \theta) \Rightarrow 2^\theta = \theta^+$).

Proof. Straight, e.g. $\mathscr{T}_\theta = ({}^{\theta >}2, \triangleleft)$. $\square_{2.4}$

Definition 2.5. Let $\mathbf{Q}$ be the set of objects $\mathbf{x}$ consisting of (below $\alpha, \beta \leqslant \lambda^+$):

(a) $\mathbb{P}_\alpha \in \mathscr{H}(\lambda^{++})$ and $I_{<\alpha}, f_\alpha \in \mathscr{H}(\lambda^{++})$ witnessed $\mathbb{P}_\alpha \in \mathbf{K}$ for $\alpha \leqslant \lambda^+$, all in $\mathscr{H}(\lambda^+)$ if $\alpha < \lambda^+$
(b) $I_\alpha \in \mathscr{H}(\lambda^+)$ and $\underset{\sim}{\mathbb{Q}}_\alpha, \underset{\sim}{g}_\alpha \in \mathscr{H}(\lambda^+)$ are $\mathbb{P}_\alpha$-names such that $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha \in \mathbf{K}$ as witnessed by $I_\alpha, \underset{\sim}{g}_\alpha$" for $\alpha < \lambda^+$
(c) $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha\colon \alpha < \lambda^+ \rangle \in \mathscr{H}(\lambda^{++})$ is an FS iteration except that:
 (∗) $\mathbb{P}_\alpha = \{p\colon p$ a finite function with domain $\subseteq \alpha$ such that if $\beta \in \mathrm{dom}(p)$ then $\underset{\sim}{g}_\beta(p(\beta)) \in \mathrm{fin}(I_\beta)$ is an object (not just a $\mathbb{P}_\beta$-name)$\}$
(d) $I_{<\alpha} = \bigcup\{I_\beta\colon \beta < \alpha\}$ is disjoint to I_α and $\mathbb{P} = \mathbb{P}_{\lambda^+} = \bigcup\{\mathbb{P}_\alpha\colon \alpha < \lambda^+\}$ and $f_\alpha(p) = \bigcup\{\underset{\sim}{g}_\alpha(p(\beta))\colon \beta \in \mathrm{dom}(p)\}$
(e) E is a club of λ^+ and for $\alpha \in S \cap E$:
 (α) $\underset{\sim}{\bar{D}}_\alpha = \langle \underset{\sim}{D}_{\alpha,s}\colon s \in \mathscr{T} \rangle$ is a $\mathbb{P}_\alpha$-name of an ultra $\mathscr{T}$-filter system (equivalently each $\underset{\sim}{\bar{D}}_{\alpha,\theta} = \underset{\sim}{\bar{D}}_\alpha \restriction \mathscr{T}_\theta$ is a $\mathbb{P}_\alpha$-name of an ultra $\mathscr{T}_\theta$-filter system), and for simplicity $\mathrm{fil}(\underset{\sim}{D}_{\alpha,s}) = \underset{\sim}{D}_{\alpha,s}$
 (β) $\langle \underset{\sim}{D}_{\beta,s}\colon \beta \in S \cap E,\ \beta \leqslant \alpha \rangle$ is $\subseteq$-increasing continuous for each $s \in \mathscr{T}$
(f) if $\alpha \in S \cap E$ <u>then</u> $\underset{\sim}{\mathbb{Q}}_\alpha$ is $\mathbb{Q}_{\underset{\sim}{D}_{\alpha,t(\alpha)}}$ see Definition 1.7 and calling the generic $\underset{\sim}{\eta}_\alpha$, we have $I_\alpha = \{0\}$, $\underset{\sim}{g}_\alpha(p) = \mathrm{tr}(p)$
(g) (α) if $\alpha \in S \cap E$ and $s, t \in \mathscr{T}$ <u>then</u> $\Vdash_{\mathbb{P}_\alpha} \mathrm{fil}(\underset{\sim}{D}_{\alpha,s}) \subseteq \mathrm{fil}(\underset{\sim}{D}_{\alpha,t})$ iff $s \leqslant_{\mathscr{T}} t$ actually follows from (e)
 (β) if $\alpha < \beta$ are from $S \cap E$ and $s \in \mathscr{T}$ <u>then</u> $\Vdash_{\mathbb{P}_\beta}$ "if $\underset{\sim}{A} \in \mathrm{id}(\underset{\sim}{\mathbf{d}}^\alpha_{t(\alpha),s})[\mathbb{P}_\alpha]$; then $\underset{\sim}{A} = \emptyset \bmod \underset{\sim}{D}_{\beta,s}$" where $\underset{\sim}{\mathbf{d}}^\alpha_{t,s} = (\underset{\sim}{D}_{\alpha,t}, \underset{\sim}{D}_{\alpha,s})$
(h) (α) if $\delta \in S_0 \cap E$ <u>then</u> $\underset{\sim}{\mathbb{Q}}_\delta$ is $\mathbb{Q}_{\mathrm{fil}(\emptyset)}$ with $\underset{\sim}{\nu}^*_\delta$ the generic
 (β) if $\delta \in S_0 \cap E$ and $C^*_\delta \subseteq E_{\mathbf{x}}$, see 2.2(4)($\varepsilon$) then $\underset{\sim}{u}_{\delta,t,n} \in \underset{\sim}{D}_{\gamma,t}$, see below, whenever $t \in \mathscr{T}, n \in \mathbb{N}$ and $\gamma \in S \cap E_{\mathbf{x}} \backslash (\delta + 1)$
 (γ) in clause (β) we let $\underset{\sim}{u}_{\delta,t,m} = \{\underset{\sim}{\eta}_{\Upsilon(\delta,t,n)}(k)\colon n \in \mathbb{N}, n \geqslant m$ and $k \geqslant \underset{\sim}{\nu}^*_\delta(n)\}$.

Discussion 2.6. 1) Later we shall use an increasing continuous sequence $\langle \mathbf{x}_\varepsilon\colon \varepsilon \leqslant \lambda \rangle$. Where and how will cofinality κ reappear? Well, we shall use $\mathbb{P}_{\delta(*)}[\mathbf{x}_\lambda]$ for some $\delta(*) \in E_{\mathbf{x}_\lambda}$ of cofinality κ. So why not replace λ^+ by κ above? We have a problem in proving the existence of a (canonical) upper bound to $\langle \mathbf{x}_\varepsilon\colon \varepsilon < \delta \rangle$, specifically in finding the $\underset{\sim}{D}_{\beta_i}$ in the proof of

Claim 2.11, i.e. completing an appropriate $\mathscr{T}$-filter system to an ultra one, e.g. in Case 3 in the proof of 2.11. To help we carry a strong induction hypothesis, see clause (i)(γ)$\bullet_2$ in $\boxdot$ there and then first find an $\mathbb{R}_{\beta_j,\lambda^+}[\mathbb{P}_{\beta_j}\bar{\mathbf{x}}]$-name, then reflect it to a β_i.

2) Note that it helps to have not only $\underaccent{\tilde}{\mathbb{Q}}_\alpha = \mathbb{Q}_{\bar{\underaccent{\tilde}{D}}}$, but possibly some related forcing notions. First in proving there is a limit, see 2.11, in proving the "reflection" discussed above lead us to use some unions. Second, using ultrapower by E_θ, see 2.13, for limit δ of cofinality θ, the ultrapower naturally leads us to use some iterations.

3) We may in 2.1 demand $\kappa \notin \Theta_1$, equivalently $\kappa < \min(\Theta)$, but let $\mathscr{T}_\kappa$ be a singleton $\{t_*\}$ and $\mathscr{T}$ is $\mathscr{T}_{\geqslant \min(\Theta_1)} \cup \mathscr{T}_\kappa$. In this case in 2.17 we get $\Vdash_{\mathbb{P}\times\mathscr{T}_*}$ "$\{\kappa\} \cup \Theta_1 \subseteq \mathrm{Sp}_\chi$ ".

Definition 2.7. 1) For $\mathbf{x} \in \mathbf{Q}$, of course we let $\bar{\mathbb{Q}}^{\mathbf{x}} = \bar{\mathbb{Q}}_{\mathbf{x}} = \bar{\mathbb{Q}}[\mathbf{x}] = \bar{\mathbb{Q}}, \mathbb{P}^{\mathbf{x}}_\alpha = \mathbb{P}_\alpha[\mathbf{x}] = \mathbb{P}_\alpha, \mathbb{P}_{\mathbf{x}} = \mathbb{P}^{\mathbf{x}} = \mathbb{P} = \mathbb{P}^{\mathbf{x}}_{\lambda^+}$, etc.

2) We define a two-place relation $\leqslant_{\mathbf{Q}}$ on $\mathbf{Q}$: $\mathbf{x} \leqslant_{\mathbf{Q}} \mathbf{y}$ iff:

(a) $(I^{\mathbf{x}}_{<\alpha}, f^{\mathbf{x}}_\alpha, \mathbb{P}^{\mathbf{x}}_\alpha) \leqslant^{\mathrm{st}}_{\mathbf{K}} (I^{\mathbf{y}}_{<\alpha}, f^{\mathbf{y}}_\alpha, \mathbb{P}^{\mathbf{y}}_\alpha)$ for $\alpha \leqslant \lambda^+$, see Definition 1.17(3)
(b) $\Vdash_{\mathbb{P}^{\mathbf{y}}_\alpha}$ "$(I^{\mathbf{x}}_\alpha, \underaccent{\tilde}{g}^{\mathbf{x}}_\alpha, \underaccent{\tilde}{\mathbb{Q}}^{\mathbf{x}}_\alpha) \leqslant^{\mathrm{wk}}_{\mathbf{K}} (I^{\mathbf{y}}_\alpha, \underaccent{\tilde}{g}^{\mathbf{y}}_\alpha, \underaccent{\tilde}{\mathbb{Q}}^{\mathbf{y}}_\alpha)$" for $\alpha < \lambda^+$, see Definition 1.17(2)
(c) $E_{\mathbf{y}} \subseteq E_{\mathbf{x}}$
(d) $\Vdash_{\mathbb{P}_\alpha[\mathbf{y}]}$ "$\underaccent{\tilde}{D}^{\mathbf{x}}_{\alpha,t(i)} \subseteq \underaccent{\tilde}{D}^{\mathbf{y}}_{\alpha,t(i)}$" for $\alpha \in S \cap E_{\mathbf{y}}$ and $t \in \mathscr{T}$
(e) $\Vdash_{\mathbb{P}_\alpha[\mathbf{y}]}$ "if $A \in ((\underaccent{\tilde}{D}^{\mathbf{x}}_{\alpha,t(i)})^+)^{\mathbf{V}[\mathbb{P}[\mathbf{x}]]}$ then $A \in (\underaccent{\tilde}{D}^{\mathbf{y}}_{\alpha,t(i)})^+$", really follows by clause (d) and 2.5(e)(α), the "ultra".

Claim 2.8. *$\mathbf{Q}$ is non-empty, in fact there is $\mathbf{x} \in \mathbf{Q}$ such that $\mathbb{P}^{\mathbf{x}}_\alpha$ has cardinality λ for $\alpha \in [1, \lambda^+)$ and in $\mathbf{V}^{\mathbb{P}^{\mathbf{x}}_1}$ we have $2^{\aleph_0} = \lambda$.*

Proof. For $i = 0$, first letting $D'_{0,s} = \emptyset$ for $s \in \mathscr{T}$, clearly $\underaccent{\tilde}{D}'_0 = \langle D'_{0,s}\colon s \in \mathscr{T}\rangle$ is a $\mathscr{T}$-filter system hence by 1.4(2) we can choose $\bar{D}_0 = \langle D_{0,s}\colon s \in \mathscr{T}\rangle$, an ultra $\mathscr{T}$-filter system (in $\mathbf{V} = \mathbf{V}^{\mathbb{P}_0}$). Second, we choose $\mathbb{Q}_i$ as adding λ Cohen reals, say $\langle \underaccent{\tilde}{\eta}^\ell_{1,\alpha}\colon \alpha < \lambda\rangle$ so $I_i = \lambda$, g_i is the identity, so $g_i(p)(\alpha) = p(\alpha) \in {}^{\omega>}2$. Third, let $\langle (s_\alpha, t_\alpha)\colon \alpha < \lambda\rangle$ be such that $s_\alpha, t_\alpha \in \mathscr{T}$ are $\leqslant_{\mathscr{T}}$-incomparable and any such pair appears.

We define a $\mathbb{P}_1$-name $\bar{\underaccent{\tilde}{D}}' = \langle \underaccent{\tilde}{D}'_t\colon t \in \mathscr{T}\rangle$ by $\underaccent{\tilde}{D}'_t = \{\eta^{-1}_{1,\alpha}\{\ell\}\colon s_\alpha \leqslant_I t \wedge \ell = 0$ or $t_\alpha \leqslant_I t \wedge \ell = 1\} \cup D_{0,t}$. Clearly $\Vdash_{\mathbb{P}_1}$ "$\bar{\underaccent{\tilde}{D}}'$ is a $\mathscr{T}$-filter system", so by 1.4(2) there is $\bar{\underaccent{\tilde}{D}}_1$ such that $\Vdash_{\mathbb{P}_1}$ "$\bar{\underaccent{\tilde}{D}}_1$ is an ultra $\mathscr{T}$-filter satisfying $\bar{\underaccent{\tilde}{D}}' \leqslant \bar{\underaccent{\tilde}{D}}_1$ hence $\bar{D}_0 \leqslant \bar{\underaccent{\tilde}{D}}_1$".

Now we shall choose $\mathbb{P}_\alpha, \bar{\underaccent{\tilde}{D}}_\alpha$ by induction on $\alpha \leqslant \lambda^+$ also for $\alpha \in \lambda\backslash S$ such that the relevant demands from Definition 2.5 hold, in particular, $\langle \mathbb{P}_\beta, \underaccent{\tilde}{\mathbb{Q}}_\gamma\colon \beta \leqslant \alpha,\ \gamma < \alpha\rangle$ is an FS iteration but $\gamma \in \mathrm{dom}(p), p \in \mathbb{P}_\beta$ implies that $\emptyset \in \mathbb{P}_\beta$ forces a value to $\mathrm{tr}(p(\gamma))$ and also $\Vdash_{\mathbb{P}_\alpha}$ "$\bar{\underaccent{\tilde}{D}}_\alpha$ is a $\mathscr{T}$-filter system such that $\bar{\underaccent{\tilde}{D}}_\beta \leqslant \bar{\underaccent{\tilde}{D}}_\alpha$ for $\beta < \alpha$ and $\bar{\underaccent{\tilde}{D}}_\alpha$ is ultra when $\alpha \notin S_0$"; recall that in Definition 2.5 $\bar{\underaccent{\tilde}{D}}_\alpha$ is defined only for $\alpha \in E \cap S$, but no harm in defining $\bar{\underaccent{\tilde}{D}}_\alpha$ in more cases. For $\alpha = 0, 1$ this was done above.

For α limit let $\mathbb{P}_\alpha = \bigcup\{\mathbb{P}_\beta\colon \beta < \alpha\}$ and $\bar{\underaccent{\tilde}{D}}'_\alpha = \langle \underaccent{\tilde}{D}'_{\alpha,t}\colon t \in \mathscr{T}_*\rangle$ where $\underaccent{\tilde}{D}'_{\alpha,t(i)} = \bigcup\{\underaccent{\tilde}{D}_{\beta,t(i)}\colon \beta < \alpha\}$. It is easy to see that $\langle \mathbb{P}_\beta\colon \beta \leqslant \alpha\rangle$ is a $\lessdot$-increasing continuous sequence of c.c.c. forcing notions and $\Vdash_{\mathbb{P}_\alpha}$ "$\bar{\underaccent{\tilde}{D}}'_\alpha$ is a $\mathscr{T}$-filter system". If $\delta \in S_0$ let $\bar{\underaccent{\tilde}{D}}_\alpha = \bar{\underaccent{\tilde}{D}}'_\alpha$, otherwise by 1.4(2) we can find $\bar{\underaccent{\tilde}{D}}_\alpha$ such that $\Vdash_{\mathbb{P}_\alpha}$ "$\bar{\underaccent{\tilde}{D}}_\alpha$ is an ultra $\mathscr{T}$-filter system and $\bar{\underaccent{\tilde}{D}}'_\alpha \leqslant \bar{\underaccent{\tilde}{D}}_\alpha$".

For $\alpha = \beta + 1$ such that $\beta \notin S \cup S_0$ let $\underaccent{\tilde}{\mathbb{Q}}_\beta$ be trivial. Now let $\mathbb{P}_\alpha = \mathbb{P}_\beta * \underaccent{\tilde}{\mathbb{Q}}_\beta$ and let $\underaccent{\tilde}{D}'_{\alpha,t}$ be $\underaccent{\tilde}{D}_{\beta,t}$. Easily $\Vdash_{\mathbb{P}_\alpha}$ "$\langle \underaccent{\tilde}{D}'_{\alpha,t}\colon t \in \mathscr{T}\rangle$ is a $\mathscr{T}$-filter system" and choose $\bar{\underaccent{\tilde}{D}}_\alpha$ as above, i.e. (a $\mathbb{P}_\alpha$-name of an) ultra $\mathscr{T}$-filter system above $\bar{\underaccent{\tilde}{D}}'_\alpha$.

Next, assume $\alpha = \beta + 1, \beta \in S$; we let $\underaccent{\tilde}{\mathbb{Q}}_\beta = \mathbb{Q}_{\underaccent{\tilde}{D}_{\beta,t(\beta)}}$ and $\mathbb{P}_\alpha = \mathbb{P}_\beta * \underaccent{\tilde}{\mathbb{Q}}_\beta$. Now for $s \in \mathscr{T}$, let $\underaccent{\tilde}{D}'_{\alpha,s} = \underaccent{\tilde}{D}_{\beta,s} \cup \{\mathbb{N}\backslash A\colon A \in \mathrm{id}_{\underaccent{\tilde}{\mathbf{d}}_{t(\beta),s}}\}$ where $t(\beta) = t_\beta$ is from 2.2(3). Note that $\Vdash_{\mathbb{P}_\beta}$ "$\mathrm{fil}(\underaccent{\tilde}{D}_{\alpha,s}) \subseteq \mathrm{fil}(\underaccent{\tilde}{D}_{\alpha,t})$ iff $s \leqslant_{\mathscr{T}} t$" by the choice of the $\underaccent{\tilde}{D}_{1,s}$'s and the $\underaccent{\tilde}{D}_{\beta,s}$'s, so the definition of $\mathrm{id}_{\underaccent{\tilde}{\mathbf{d}}_{t(\beta),s}}$ depend on the truth value of $t(\beta) \leqslant_I s$.

Now (pedantically working in $\mathbf{V}^{\mathbb{P}_\beta}$):

- $\underaccent{\tilde}{D}'_{\alpha,s} \subseteq [\mathbb{N}]^{\aleph_0}$ by its definition
- $\underaccent{\tilde}{D}_{\alpha,s} \subseteq \underaccent{\tilde}{D}'_{\alpha,s}$, by 1.12
- $\emptyset \notin \mathrm{fil}(D'_{\alpha,s})$ by 1.12
- if $A \in (D^+_{\alpha,s})^{\mathbf{V}[\mathbb{P}_\beta]}$ then $A \in ((D'_{\alpha,s})^+)^{\mathbf{V}[\mathbb{P}_{\beta+1}]}$ by 1.12
- $s \leqslant_I t \Rightarrow \underaccent{\tilde}{D}'_{\alpha,s} \subseteq \underaccent{\tilde}{D}'_{\alpha,t}$ by 1.13 and the choice of the $D'_{\alpha,t}$'s.

We continue as in the previous case.

Lastly, assume $\alpha = \beta + 1, \beta \in S_0$ and we shall define for α. We let $\underaccent{\tilde}{\mathbb{Q}}_\beta = \mathbb{Q}_{\mathrm{fil}(\emptyset)}$ in $\mathbf{V}^{\mathbb{P}_\beta}$ and so $\underaccent{\tilde}{\nu}^*_\beta$ is defined as the generic and $\mathbb{P}_{\beta+1} = \mathbb{P}_\beta * \underaccent{\tilde}{\mathbb{Q}}_\beta$. Note that $\underaccent{\tilde}{u}_{\beta,t,n}$ is well defined (see clause (h) of Definition 2.5). By Claim 2.9 below letting $\underaccent{\tilde}{D}'_{\alpha,t} = \underaccent{\tilde}{D}_{\beta,t} \cup \{\underaccent{\tilde}{u}_{\beta,s,n}\colon n \in \mathbb{N}$ and $s \in \mathscr{T}$ satisfies $s \leqslant_{\mathscr{T}} t\}$ we have $\bar{\underaccent{\tilde}{D}}'_\alpha = \langle \underaccent{\tilde}{D}'_{\alpha,t}\colon t \in \mathscr{T}\rangle$ is a $\mathbb{P}_\beta$-name of a $\mathscr{T}$-filter system above $\bar{\underaccent{\tilde}{D}}_\beta$ and let $\bar{\underaccent{\tilde}{D}}_\alpha$ be (a $\mathbb{P}_\alpha$-name of) an ultra $\mathscr{T}$-filter system above $\bar{\underaccent{\tilde}{D}}'_\alpha$.

Let $I_\alpha = \{\alpha\}$ for $\alpha < \lambda^+$, $I_{<\alpha} = \alpha$ for $\alpha \leqslant \lambda^+$ and if $\alpha \in S \cup S_0$ then we let $\Vdash_{\mathbb{P}_\alpha}$ "if $p \in \underaccent{\tilde}{\mathbb{Q}}_\alpha$ then $\underaccent{\tilde}{g}_\alpha(p)$ is $\mathrm{tr}(p)$, the trunk" and if $\alpha \in \lambda^+\backslash(S \cup S_0)$ then $\underaccent{\tilde}{g}_\alpha(p) = 0$.

Naturally, we define $\mathbf{x}$ by: $\mathbb{P}^{\mathbf{x}}_\beta = \mathbb{P}_\beta$, $\underaccent{\tilde}{\mathbb{Q}}^{\mathbf{x}}_\alpha = \underaccent{\tilde}{\mathbb{Q}}_\alpha$, $E_{\mathbf{x}} = \lambda$, $I^{\mathbf{x}}_\alpha = I_\alpha$, $I^{\mathbf{x}}_{<\beta} = I_{<\beta}$, $\underaccent{\tilde}{g}^{\mathbf{x}}_\alpha = \underaccent{\tilde}{g}_\alpha$ for $\alpha < \lambda^+, \beta \leqslant \lambda^+$ (and so $f^{\mathbf{x}}_\alpha$ is defined), $\bar{\underaccent{\tilde}{D}}^{\mathbf{x}}_\gamma = \underaccent{\tilde}{D}_\gamma$ for $\gamma \in S$, $\beta \leqslant \lambda^+$, $\alpha < \lambda$. It is easily to check that $\mathbf{x} \in \mathbf{Q}$ is as required. $\square_{2.8}$

Claim 2.9. *If* (A) *then* (B) *where*

(A) (a) $\delta \in S_0$
  (b) $\mathbb{P}_\alpha(\alpha \leqslant \delta)$, $\mathbb{Q}_\alpha(\alpha \leqslant \delta)$, $E \subseteq \delta$, *etc., are as in Definition* 2.5 *except that all is up to* δ
  (c) $\underset{\sim}{\mathbb{Q}}_\delta$, $\underset{\sim}{\nu}^*_\delta$, $\underset{\sim}{u}_{\delta,t}$ *are as in clause* (h) *of Definition* 2.5
  (d) $\underset{\sim}{D}'_{\delta,t} := \bigcup\{\underset{\sim}{D}_{\alpha,t}\colon \alpha \in S \cap E\} \cup \{\underset{\sim}{u}_{\delta,t,n}\colon n \in \mathbb{N}$ *and* $s \in \mathscr{T}$ *satisfies* $s \leqslant_{\mathscr{T}} t\}$ *so a* $\mathbb{P}_\delta * \underset{\sim}{\mathbb{Q}}_{\mathrm{fil}(\emptyset)}$*-name*

(B) (a) $\Vdash_{\mathbb{P}_\delta * \underset{\sim}{\mathbb{Q}}_{\mathrm{fil}(\emptyset)}}$ *"*$\langle \underset{\sim}{D}'_{\delta,t}\colon t \in \mathscr{T}\rangle$ *is a* $\mathscr{T}$*-filter system"*
  (b) $\Vdash_{\mathbb{P}_\delta * \underset{\sim}{\mathbb{Q}}_{\mathrm{fil}(\emptyset)}}$ "$\mathrm{fil}(\underset{\sim}{D}'_{\delta,t}) = \mathrm{fil}(\{\underset{\sim}{u}_{\delta,s,n}\colon s \leqslant_{\mathscr{T}} t$ *and* $n \in \mathbb{N}\})$"
  (c) $\Vdash_{\mathbb{P}_\delta * \underset{\sim}{\mathbb{Q}}_{\mathrm{fil}(\emptyset)}}$ *"if* $t \in \mathscr{T}$ *and* $A \in \bigcup\{D_{\alpha,t}\colon \alpha \in \delta \cap S\}$ *then* $\underset{\sim}{u}_{\delta,t,n} \subseteq^* A$ *for every large enough n".*

Proof. Straight; the point is $\Vdash_{\mathbb{P}_\delta * \underset{\sim}{\mathbb{Q}}_{\mathrm{fil}(\emptyset)}}$ "$\emptyset \notin \mathrm{fil}(\underset{\sim}{D}'_{\delta,t})$" for $t \in \mathscr{T}$, which holds as

$(*)_1$ if $A \in \underset{\sim}{D}_{\Upsilon(\delta,t,n)}$ <u>then</u> for every large enough k, $\underset{\sim}{\eta}_{\Upsilon(\delta,t,n)}(k) \in A$
$(*)_2$ if $A \in \underset{\sim}{D}^+_{\Upsilon(\delta,t,n)}$ in $\mathbf{V}^{\mathbb{P}_\delta}$ <u>then</u> for infinitely many k, $\underset{\sim}{\eta}_{\Upsilon(\delta,t,n)}(k) \in A$
$(*)_3$ $\underset{\sim}{\nu}^*_\delta$ is a dominating real. $\square_{2.9}$

Observation 2.10. 1) $\leqslant_{\mathbf{Q}}$ partially orders $\mathbf{Q}$.
2) $\mathbb{P}^{\mathbf{x}}_\alpha$ satisfies the c.c.c. and even is locally $\aleph_1$-centered[3] when $\mathbf{x} \in \mathbf{Q}$ and $\alpha \leqslant \lambda^+$.

Proof. Easy. $\square_{2.10}$

Claim 2.11 *(The upper bound existence claim). If* $\langle \mathbf{x}_\varepsilon\colon \varepsilon < \delta\rangle$ *is* $\leqslant_{\mathbf{Q}}$*-increasing and* δ *is a limit ordinal* $< \lambda^+$ <u>*then*</u> *there is* $\mathbf{x}_\delta$ *which is a canonical limit of* $\langle \mathbf{x}_\varepsilon\colon \varepsilon < \delta\rangle$*, see below.*

Definition 2.12. We say $\mathbf{x} = \mathbf{x}_\delta$ is a canonical limit of $\bar{\mathbf{x}} = \langle \mathbf{x}_\varepsilon\colon \varepsilon < \delta\rangle$ <u>when</u> $\bar{\mathbf{x}}$ is $\leqslant_{\mathbf{Q}}$-increasing, δ is a limit ordinal $< \lambda^+$ and (for every $\alpha < \lambda^+$):

(a) $\mathbf{x}_\delta \in \mathbf{Q}$
(b) $\mathbf{x}_\varepsilon \leqslant_{\mathbf{Q}} \mathbf{x}_\delta$ for $\varepsilon < \delta$ and $E_{\mathbf{x}_\delta} \subseteq \bigcap\{E_{\mathbf{x}_\varepsilon}\colon \varepsilon < \delta\}$
(c) $I_\alpha[\mathbf{x}_\delta] = \bigcup\{I_\alpha[\mathbf{x}_\varepsilon]\colon \varepsilon < \delta\}$
(d) if δ has uncountable cofinality <u>then</u>
  (α) $\mathbb{P}^{\mathbf{x}_\delta}_\alpha = \bigcup\{\mathbb{P}^{\mathbf{x}_\varepsilon}_\alpha\colon \varepsilon < \delta\}$
  (β) $\Vdash_{\mathbb{P}^{\mathbf{x}_\delta}_\alpha}$ "$\underset{\sim}{D}^{\mathbf{x}_\delta}_{\alpha,t} = \bigcup\{\underset{\sim}{D}^{\mathbf{x}_\varepsilon}_{\alpha,t}\colon \varepsilon < \delta\}$" for $t \in \mathscr{T}$ if $\alpha \in E_{\mathbf{x}_\delta} \cap S$
  (γ) $\underset{\sim}{\mathbb{Q}}^{\mathbf{x}_\delta}_\alpha = \bigcup\{\underset{\sim}{\mathbb{Q}}^{\mathbf{x}_\varepsilon}_\alpha\colon \varepsilon < \delta\}$
  (δ) $\underset{\sim}{g}^{\mathbf{x}_\varepsilon}_\alpha = \bigcup\{\underset{\sim}{g}^{\mathbf{x}_\varepsilon}_\alpha\colon \varepsilon < \delta\}$.
(e) if δ has cofinality $\aleph_0$, then
  (α) if $\alpha \in \lambda^+ \backslash (S \cap E_{\mathbf{x}_\delta}) \backslash (S_0 \cap E_{\mathbf{x}_\delta})$ or $\alpha \in S_0 \wedge C^*_\alpha \nsubseteq E_{\mathbf{x}_\delta}$ then $\Vdash_{\mathbb{P}_\alpha[\mathbf{x}_\delta]}$ "$\underset{\sim}{\mathbb{Q}}_\alpha[\mathbf{x}_\delta] = \bigcup\{\underset{\sim}{\mathbb{Q}}_\alpha[\mathbf{x}_\varepsilon]\colon \varepsilon < \delta\}$ and similarly $\underset{\sim}{g}_\alpha[\mathbf{x}_\delta] = \bigcup\{\underset{\sim}{g}_\alpha[\mathbf{x}_\varepsilon]\colon \varepsilon < \delta\}$
  (β) if $\alpha \in S \cap E_{\mathbf{x}_\delta}$ then $\Vdash_{\mathbb{P}_\alpha[\mathbf{x}_\delta]}$ "$\underset{\sim}{D}_{\alpha,t}[\mathbf{x}_\delta] \supseteq \bigcup\{\underset{\sim}{D}_{\alpha,t}[\mathbf{x}_\varepsilon]\colon \varepsilon < \delta\}$"
(f) in fact $|\mathbb{P}^{\mathbf{x}_\delta}_\alpha| \leqslant (\Sigma\{|\mathbb{P}^{\mathbf{x}_\varepsilon}_\alpha|\colon \varepsilon < \delta\})^{\aleph_0}$.

Proof. Let

$\boxplus_0$ (a) $I_\alpha = \bigcup\{I_\alpha[\mathbf{x}_\varepsilon]\colon \varepsilon < \delta\}$ for $\alpha < \lambda^+$
  (b) $I_{<\alpha} = \bigcup\{I_\beta\colon \beta < \alpha\}$ for $\alpha \leqslant \lambda^+$
  (c) $E := \bigcap\{E[\mathbf{x}_\varepsilon]\colon \varepsilon < \delta\}$.

So $E \subseteq \bigcap\{E[\mathbf{x}_\varepsilon]\colon \varepsilon < \delta\}$ and clearly E is a club of λ^+ (but in general this will not be $E[\mathbf{x}_\delta]$). If $\beta \leqslant \gamma \leqslant \lambda^+$ and $\mathbb{Q}$ satisfies $\varepsilon < \delta \Rightarrow \mathbb{P}_\beta[\mathbf{x}_\varepsilon] \lessdot \mathbb{Q}$ and for transparency $q \in \mathbb{Q} \Rightarrow \emptyset \leqslant_{\mathbb{Q}} q$ then $\mathbb{R} = \mathbb{R}_{\beta,\gamma}[\mathbb{Q}, \bar{\mathbf{x}}]$ is defined as follows:

$\boxplus_1$ (a) $p \in \mathbb{R}$ iff $p = (p_1, p_2)$ and some pair (ε, p_0) witness it which means $\varepsilon < \delta$ and $p_0 \in \mathbb{P}_\beta[\mathbf{x}_\varepsilon]$, $p_1 \in \mathbb{P}_\gamma[\mathbf{x}_\varepsilon]$, $p_2 \in \mathbb{Q}$ and one of the following occurs
    (α) $p_1 = \emptyset$ or $p_2 = \emptyset$ recalling clause (c) of 2.5
    (β) $p_0 \Vdash_{\mathbb{P}_\beta[\mathbf{x}_\varepsilon]}$ "$p_1 \in \mathbb{P}_\gamma[\mathbf{x}_\varepsilon]/\mathbb{P}_\beta[\mathbf{x}_\varepsilon]$ and $p_2 \in \mathbb{Q}/\mathbb{P}_\beta[\mathbf{x}_\varepsilon]$"
  (b) for $p \in \mathbb{R}$ let $\varepsilon(p)$ be the minimal $\varepsilon < \delta$ such that (ε, p_0) witness $p \in \mathbb{R}$ for some p_0
  (c) $\mathbb{R} \models$ "$p \leqslant q$" iff letting $\varepsilon = \max\{\varepsilon(p), \varepsilon(q)\}$ we have $\mathbb{P}_\gamma[\mathbf{x}_\varepsilon] \models$ "$p_1 \leqslant q_1$" and $\mathbb{Q} \models$ "$p_2 \leqslant q_2$".

[3] Meaning that any $\aleph_1$ elements can be divided to $\aleph_0$ sets such that any finitely many members of one sets has a common upper bound.

We note that:

$\boxplus_2$ (a) $\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$ is a partial order
(b) above $\mathbb{R}'_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$ is a dense subset of $\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$ where $\mathbb{R}'_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$ is defined like $\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$ when in $\boxplus_1(a)$ we omit subclause (α).

[Why? Clause (a) by $\boxplus_3$ below and clause (b) is easy.]
So below we may ignore the difference between $\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$ and $\mathbb{R}'_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$

$\boxplus_3$ for $(\beta,\gamma,\mathbb{Q})$ as above; if (ε,p_0) is a witness for $p=(p_1,p_2)\in\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$ and $\zeta\in(\varepsilon,\delta)$ <u>then</u> for some $q_0\in\mathbb{P}_\beta[\mathbf{x}_\zeta]$ the pair (ζ,q_0) is a witness for $(p_1,p_2)\in\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$.

[Why? As we can increase p_0 in $\mathbb{P}_\beta[\mathbf{x}_\varepsilon]$, without loss of generality $(p_1{\restriction}\beta)\leqslant p_0$, where on $\restriction$ recall Definition 2.5, clause (c). As (ε,p_0) is a witness for $(p_1,p_2)\in\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$ necessarily p_0,p_2 are compatible in $\mathbb{Q}$ hence they have a common upper bound $q_2\in\mathbb{Q}$. As $\mathbb{P}_\beta[\mathbf{x}_\zeta]\lessdot\mathbb{Q}$, there is $q_0\in\mathbb{P}_\beta[\mathbf{x}_\zeta]$ such that $q_0\leqslant q\in\mathbb{P}_\beta[\mathbf{x}_\zeta]\Rightarrow q,q_2$ are compatible in $\mathbb{Q}$. As we can increase q_0 in $\mathbb{P}_\beta[\mathbf{x}_\zeta]$ and $p_0\leqslant q_2$ without loss of generality $p_0\leqslant q_0$ but $(p_1{\restriction}\beta)\leqslant p_0$ hence $(p_1{\restriction}\beta)\leqslant q_0$. As $\mathbf{x}_\varepsilon\leqslant\mathbf{x}_\zeta$ and $\langle\mathbb{P}_\alpha[\mathbf{x}_\zeta],\tilde{\mathbb{Q}}_\alpha[\mathbf{x}_\zeta]\colon\alpha<\lambda^+\rangle$ is FS iteration and $p_1\in\mathbb{P}_\gamma[\mathbf{x}_\varepsilon]\lessdot\mathbb{P}_\gamma[\mathbf{x}_\zeta]$, clearly $q_0\leqslant q\in\mathbb{P}_\beta[\mathbf{x}_\zeta]\Rightarrow q,p_1$ are compatible. So clearly (ζ,q_0) is a witness for $p\in\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$ as required in $\boxplus_3$.]

$\boxplus_4$ if $\beta,\gamma,\mathbb{Q}$ are as above and $\gamma\leqslant\gamma(1)\leqslant\lambda^+$ <u>then</u> $\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]\lessdot\mathbb{R}_{\beta,\gamma(1)}[\mathbb{Q},\bar{\mathbf{x}}]$.

[Why? We check the conditions from Definition 1.1(3), the second alternative. First, if $p=(p_1,p_2)\in\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$ we shall prove $p\in\mathbb{R}_{\beta,\gamma(1)}[\mathbb{Q},\bar{\mathbf{x}}]$; as $p\in\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$, some (ε,p_0) witness it, easily it witnesses $p\in\mathbb{R}_{\beta,\gamma(1)}[\mathbb{Q},\bar{\mathbf{x}}]$ as $\mathbb{P}_\gamma[\mathbf{x}_\varepsilon]\subseteq\mathbb{P}_{\gamma(1)}[\mathbf{x}_\varepsilon]$.

Second, assume $\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]\models$ "$p\leqslant q$" and we should prove $\mathbb{R}_{\beta,\gamma(1)}[\mathbb{Q},\bar{\mathbf{x}}]\models$ "$p\leqslant q$", this is obvious by the definition of the orders for those forcing notions. Together $\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]\subseteq\mathbb{R}_{\beta,\gamma(1)}[\mathbb{Q},\bar{\mathbf{x}}]$.

Third, we should prove $\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]\subseteq_{\mathrm{ic}}\mathbb{R}_{\beta,\gamma(1)}[\mathbb{Q},\bar{\mathbf{x}}]$ so assume $p,q\in\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$ has a common upper bound $r=(r_1,r_2)$ in $\mathbb{R}_{\beta,\gamma(1)}[\mathbb{Q},\bar{\mathbf{x}}]$. Now easily $(r_1{\restriction}\gamma,r_2)$ is a common upper bound of p,q in $\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$ as required.

Fourth, for $p\in\mathbb{R}_{\beta,\gamma(1)}[\mathbb{Q},\bar{\mathbf{x}}]$ we should find $q\in\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$ such that if $\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]\models$ "$q\leqslant q^*$" then q^*,p are compatible in $\mathbb{R}_{\beta,\gamma(1)}[\mathbb{Q},\bar{\mathbf{x}}]$.

Now let $p=(p_1,p_2)\in\mathbb{R}_{\beta,\gamma(1)}[\mathbb{Q},\bar{\mathbf{x}}]$ and let (ε,p_0) witness it; without loss of generality $\mathbb{P}_\beta[\mathbf{x}_\varepsilon]\models$ "$(p_1{\restriction}\beta)\leqslant p_0$".

Let $q_1=p_1{\restriction}\gamma\in\mathbb{P}_\gamma[\mathbf{x}_\varepsilon]$, now $q:=(q_1,p_2)$ satisfies

- $q\in\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$.

Why? The pair (ε,p_0) witness it because if $p_0\leqslant q'\in\mathbb{P}_\beta[\mathbf{x}_\varepsilon]$ <u>then</u> first p_1,q' has a common upper bound $r\in\mathbb{P}_{\gamma(1)}[\mathbf{x}_\varepsilon]$ hence $r{\restriction}\gamma\in\mathbb{P}_\gamma[\mathbf{x}_\varepsilon]$ is a common upper bound of q',q_1; second q',p_2 has a common upper bound in $\mathbb{Q}$ as (ε,p_0) witness (p_1,p_2). So indeed (ε,p_0) witness $q=(q_1,p_2)\in\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$.

- If $q\leqslant q^*\in\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$ <u>then</u> q^*,p are compatible in $\mathbb{P}_{\gamma(1)}[\mathbf{x}_\varepsilon]$.

Why? Let $q^*=(q_1^*,q_2^*)$ and let $r_1=(p_1{\restriction}[\gamma,\gamma(1)))\cup q_1^*$, easily $(r_1,q_2^*)\in\mathbb{R}_{\beta,\gamma(1)}[\mathbb{Q},\bar{\mathbf{x}}]$ is a common upper bound of q^*,p.

This finishes checking the last demand for $\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]\lessdot\mathbb{R}_{\beta,\gamma(1)}[\mathbb{Q},\bar{\mathbf{x}}]$ so $\boxplus_4$ holds.]

$\boxplus_5$ if $\mathbb{Q}$ satisfies the c.c.c. then $\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$ satisfies the c.c.c.

[Why? Let $p_i=(p_{1,i},p_{2,i})\in\mathbb{R}_{\beta,\gamma}[\mathbb{Q},\bar{\mathbf{x}}]$ for $i<\aleph_1$. Let $(\varepsilon_i,p_{0,i})$ be a witness for $(p_{1,i},p_{2,i})$. As before let $q_i\in\mathbb{Q}$ be such that $p_{0,0},p_{1,i}{\restriction}\beta,p_{2,i}$ are below it.

We can find an uncountable S such that $\langle f_\gamma[\mathbf{x}_{\varepsilon_i}](p_{1,i})\colon i\in S\rangle$ are pairwise compatible functions and $\langle\varepsilon_i\colon i\in S\rangle$ is non-decreasing. As $\mathbb{Q}$ satisfies the c.c.c., for some $i<j$ from S there is a common upper bound $q\in\mathbb{Q}$ of q_i,q_j; let $\{\beta_\ell\colon\ell<n\}$ list in increasing order $\{\beta\}\cup\mathrm{dom}(p_{1,i})\cup\mathrm{dom}(p_{1,j})\backslash\beta$ and let $\beta_n=\gamma$.

By induction on $\ell\leqslant n$ we choose $r_\ell\in\mathbb{P}_{\beta_\ell}[\mathbf{x}_{\varepsilon_j}]$ such that:

- if $\ell=0$ so $\beta_\ell=\beta$ then $r_0\leqslant r\in\mathbb{P}_\beta[\mathbf{x}_{\varepsilon_j}]\Rightarrow r,q$ are compatible in $\mathbb{Q}$
- if $\ell=m+1$ then $r_m\leqslant r_\ell$
- $\mathbb{P}_{\beta_\ell}[\mathbf{x}_{\varepsilon_j}]\models$ "$(p_{1,i}{\restriction}\beta_\ell)\leqslant r_\ell$ and $(p_{1,j}{\restriction}\beta_\ell)\leqslant r_\ell$".

For $\ell=0$ use $q\in\mathbb{Q}$ and $\mathbb{P}_\beta[\mathbf{x}_{\varepsilon_j}]\lessdot\mathbb{Q}$. For $\ell=m+1$, we shall choose $r_\ell\in\mathbb{P}_{\beta_m+1}[\mathbf{x}_{\varepsilon_i}]$ as follows: if $\beta_\ell\notin\mathrm{dom}(p_{1,i})$ then $r_\ell=r_m\cup\{(\beta_\ell,p_{1,j}(\beta_\ell))\}$; if $\beta_\ell\notin\mathrm{dom}(p_{1,j})$ similarly; otherwise, i.e. if $\beta_\ell\in\mathrm{dom}(p_{1,i})\cap\mathrm{dom}(p_{1,j})$ use the demands on $\underset{\sim}{g}_{\beta_\ell}$ recalling $(*)$ of clause (c) and end of clause (d) of Definition 2.5.

Having carried the induction, (r_m, q) is well defined. Now let $r_* \in \mathbb{P}_\beta[\mathbf{x}_{\varepsilon_j}]$ be above r_0 such that $r_* \leqslant r \in \mathbb{P}_\beta[\mathbf{x}_{\varepsilon_j}] \Rightarrow r_m, r$ are compatible. Also $r_* \leqslant r \in \mathbb{P}_\beta[\mathbf{x}_{\varepsilon_j}] \Rightarrow r_0 \leqslant r \in \mathbb{P}_\beta[\mathbf{x}_{\varepsilon_j}] \Rightarrow r, q$ are compatible in $\mathbb{Q}$. So (ε_j, r_*) witness $(r_m, q) \in \mathbb{R}_{\beta,\gamma}[\mathbb{Q}, \bar{\mathbf{x}}]$ and easily (r_m, q) is above $p_i = (p_{1,i}, p_{2,i})$ and above $p_j = (p_{1,j}, p_{2,j})$, so $\boxplus_5$ holds indeed.]

$\boxplus_6$ for $\beta, \gamma, \mathbb{Q}$ as above, $\mathbb{Q} \lessdot \mathbb{R}_{\beta,\gamma}[\mathbb{Q}, \bar{\mathbf{x}}]$ when we identify $p_2 \in \mathbb{Q}$ with $(\emptyset, p_2)$.

[Why? Again, first $p \in \mathbb{Q} \Rightarrow p \in \mathbb{R}_{\beta,\gamma}[\mathbb{Q}, \bar{\mathbf{x}}]$ by the identification, and for $p, q \in \mathbb{Q}$ we have $\mathbb{Q} \models$ "$p \leqslant q$" $\Leftrightarrow \mathbb{R}_{\beta,\gamma}[\mathbb{Q}, \bar{\mathbf{x}}] \models$ "$p \leqslant q$" by the definition of the order of $\mathbb{R}_{\beta,\gamma}[\mathbb{Q}, \bar{\mathbf{x}}]$. So $\mathbb{Q} \subseteq \mathbb{R}_{\beta,\gamma}[\mathbb{Q}, \bar{\mathbf{x}}]$ holds, moreover $\mathbb{Q} \subseteq_{\mathrm{ic}} \mathbb{R}_{\beta,\gamma}[\mathbb{Q}, \bar{\mathbf{x}}]$ by the definition of the order.

Lastly, let $q \in \mathbb{R}_{\beta,\gamma}[\mathbb{Q}, \bar{\mathbf{x}}]$, so by $\boxplus_2$ without loss of generality $q = (q_1, q_2) \in \mathbb{R}'_{\beta,\gamma}[\mathbb{Q}, \bar{\mathbf{x}}]$ and we shall find $p \in \mathbb{Q}$ such that $p \leqslant p' \in \mathbb{Q} \Rightarrow p', (q_1, q_2)$ are compatible.

Let $p = q_2$, i.e. $(\emptyset, q_2)$, and the rest should be clear.]

$\boxplus_7$ for $\beta, \gamma, \mathbb{Q}$ as above we have $\mathbb{P}_\gamma[\mathbf{x}_\varepsilon] \lessdot \mathbb{R}_{\beta,\gamma}[\mathbb{Q}, \bar{\mathbf{x}}]$ when we identify $p_1 \in \mathbb{P}_\gamma[\mathbf{x}_\varepsilon]$ with $(p_1, \emptyset)$.

[Why? Similarly.]

* * *

Now by induction on $i \leqslant \lambda^+$ we choose β_i and $\mathbb{P}_\alpha, f_\alpha$ (when $\alpha \leqslant \beta_i$ and $j < i \Rightarrow \beta_j < \alpha$), $\underset{\sim}{\mathbb{Q}}_\alpha, \underset{\sim}{g}_\alpha$ (when $\alpha < \beta_i$ and $j < i \Rightarrow \beta_j \leqslant \alpha$) and[4] also $\underset{\sim}{\bar{D}}_{\beta_i}$ (when $\beta_i \in S$) such that

$\boxdot$ the relevant parts of clauses (a)–(e) of Definition 2.12 and of the definition of $\mathbf{x}_\delta \in \mathbf{Q}$ holds, in particular (all when defined):
- (a) $\mathbb{P}_\alpha \in \mathscr{H}(\lambda^+)$ is a c.c.c. forcing notion
- (b) (α) $\mathbb{P}^{\mathbf{x}_\varepsilon}_\alpha \lessdot \mathbb{P}_\alpha$ and $\mathbb{P}_{\mathbf{x}_\varepsilon} \cap \mathbb{P}_\alpha = \mathbb{P}^{\mathbf{x}_\varepsilon}_\alpha$ for $\varepsilon < \delta$
  - (β) $(I_{<\alpha}[\mathbf{x}_\varepsilon], f_\alpha[\mathbf{x}_\varepsilon], \mathbb{P}_\alpha[\mathbf{x}_\varepsilon]) \leqslant^{\mathrm{st}}_{\mathbf{K}} (I_{<\alpha}, f_\alpha, \mathbb{P}_\alpha)$
- (c) $\underset{\sim}{\bar{D}}_{\beta_i}$ is a $\mathbb{P}_{\beta_i}$-name of an I-filter system; ultra when $\beta_i \in S$; see (i)$(\gamma)\bullet_1$
- (d) if $\beta_i \in S, \varepsilon < \delta$ and $t \in \mathscr{T}$ then $\Vdash_{\mathbb{P}_{\beta_i}}$ "$\underset{\sim}{D}^{\mathbf{x}_\varepsilon}_{\beta_i,t} \subseteq \underset{\sim}{D}_{\beta_i,t}$"
- (e) $\langle \mathbb{P}_\alpha\colon \alpha \leqslant \beta_i \rangle$ is $\lessdot$-increasing continuous
- (f) if $\beta = \alpha + 1$ then $\mathbb{P}_\gamma = \mathbb{P}_\alpha * \underset{\sim}{\mathbb{Q}}_\alpha$, in fact, $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\alpha\colon \beta \leqslant \beta_i,\ \alpha < \beta_i \rangle$ is as in clause (c) of Definition 2.5
- (g) if $\neg(\exists j)$ $(\alpha = \beta_j \in S)$ then $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha = \bigcup\{\underset{\sim}{\mathbb{Q}}_\alpha[\mathbf{x}_\varepsilon]\colon \varepsilon < \delta\}$, $\underset{\sim}{g}_\alpha = \bigcup\{\underset{\sim}{g}_\alpha[\mathbf{x}_\varepsilon]\colon \varepsilon < \delta\}$"; note that $\underset{\sim}{\mathbb{Q}}_\alpha[\mathbf{x}_\varepsilon], \underset{\sim}{g}_\alpha[\mathbf{x}_\varepsilon]$ are $\mathbb{P}^{\mathbf{x}_\varepsilon}_\alpha$-names hence $\mathbb{P}_\alpha$-name by clause (b) and $\Vdash_{\mathbb{P}_\alpha}$ "$(I_\alpha[\mathbf{x}_\varepsilon], f_\alpha[\mathbf{x}_\varepsilon], \underset{\sim}{\mathbb{Q}}_\alpha[\mathbf{x}_\varepsilon]) \leqslant^{\mathrm{wk}}_{\mathbf{K}} (I_\alpha, f_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha)$
- (h) (α) if $j < i$ then $\Vdash_{\mathbb{P}_{\beta_i}}$ "$\underset{\sim}{\bar{D}}_{\beta_j} \leqslant \underset{\sim}{\bar{D}}_{\beta_i}$"
  - (β) if i is a limit ordinal and $t \in \mathscr{T}$ <u>then</u> $\Vdash_{\mathbb{P}_{\beta_i}}$ "$\underset{\sim}{D}_{\beta_i,t} = \bigcup\{\underset{\sim}{D}_{\beta_j,t}\colon j < i\}$
- (i) (α) $\langle \beta_j\colon j \leqslant i \rangle$ is increasing continuous
  - (β) if $i = 0$ then $\beta_i = 0$
  - (γ) if $i = j + 1$ then
    - $\bullet_1$ $\beta_i \in S \cap E$
    - $\bullet_2$ if $\gamma \in [\beta_i, \lambda^+] \wedge \gamma \in (S \cap E) \cup \{\lambda^+\}$ and $t \in \mathscr{T}$, <u>then</u> $\Vdash_{\mathbb{R}_{\beta_i,\gamma}[\mathbb{P}_{\beta_i}, \bar{\mathbf{x}}]}$ "$\emptyset \notin \mathrm{fil}(\bigcup\{\underset{\sim}{D}_{\gamma,t}[\mathbf{x}_\varepsilon]\colon \varepsilon < \delta\} \cup \underset{\sim}{D}_{\beta_i,t})$"
    - $\bullet_3$ if $\beta_j \in S \cap E$ then clause (g) of Definition 2.5 holds
    - $\bullet_4$ if $\beta_j \in S_0$ and $C^*_{\beta_j} \subseteq \{\beta_\iota\colon \iota < j\}$ <u>then</u> $\underset{\sim}{\mathbb{Q}}_{\beta_j} = \underset{\sim}{\mathbb{Q}}_{\mathrm{fil}(\emptyset)}$, and so the relevant case of clause $(h)(\beta)$ of Definition 2.5 holds
  - (δ) if i is a limit ordinal, $\gamma \in (\beta_i, \lambda^+] \wedge \gamma \in (S \cap E) \cup \{\lambda^+\}$ and $t \in \mathscr{T}$ <u>then</u> $\Vdash_{\mathbb{R}_{\beta_i,\gamma}[\mathbb{P}_{\beta_i}, \bar{\mathbf{x}}]}$ "$\emptyset \notin \mathrm{fil}(\bigcup\{\underset{\sim}{D}_{\gamma,t}[\mathbf{x}_\varepsilon]\colon \varepsilon < \delta\} \cup \bigcup\{D_{\alpha,t}\colon \alpha < \beta_i\})$".

Note that as $\underset{\sim}{\bar{D}}_\alpha$ (when $(\exists j \leqslant i)(\alpha = \beta_j \in S \vee j = 0)$) is an ultra $\mathscr{T}$-filter system, we do not have to bother proving $A \in (\underset{\sim}{D}^+_{\alpha,s}[\underset{\sim}{G}_{\mathbb{P}_\alpha}]) \Rightarrow A \in (\underset{\sim}{D}^+_{\beta,s}[\underset{\sim}{G}_{\mathbb{P}_\beta}])$ (when $\alpha < \beta$ are from $\{\beta_j\colon j \leqslant i, \beta_j \in S\}$).

Also

$(*)_1$ if $t \in \mathscr{T}$, $\varepsilon < \delta$, $\beta \leqslant \beta_i$ and $\beta \in S \cap E_{\mathbf{x}_\varepsilon}$ then $\Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{D}^{\mathbf{x}_\varepsilon}_{\beta,t} \subseteq \underset{\sim}{D}_{\beta_i,t}$".

[Why? This follows from clause (i) of $\boxdot$.]

Let us carry the induction, this clearly suffices.

Case 1: $i = 0$.

Trivial.

[4] So we define some $\underset{\sim}{\bar{D}}_{\beta_i}$ not used in $\mathbf{x}_\delta$.

Case 2: i is a limit ordinal.

Let $\beta = \beta_i$ be $\bigcup\{\beta_j\colon\ j < i\}$, clearly $\langle \beta_j\colon\ j \leqslant i\rangle$ is increasing continuous and $\beta_i \in E$. Below ε vary on δ.

Let $\mathbb{P}_\beta = \bigcup\{\mathbb{P}_\alpha\colon\ \alpha < \delta\}$ and $f_\beta = \bigcup\{f_\alpha\colon\ \alpha < \beta\}$ and from $\boxplus_0$ recall $I_{<\beta} = \bigcup\{I_\alpha\colon\ \alpha < \beta\}$. Clearly $\mathbb{P}_\beta \in \mathbf{K}$ as witnessed by $(I_{<\beta}, f_\beta)$ and $\alpha < \beta \Rightarrow \mathbb{P}_\alpha \lessdot \mathbb{P}_\beta$. Note that $\mathbb{P}_\beta$ satisfies the c.c.c. as $\langle \mathbb{P}_\alpha\colon\ \alpha < \beta\rangle$ is $\lessdot$-increasing continuous and the induction hypothesis; alternatively using f_α.

Now

$(*)_2$ $\mathbb{P}_\beta[\mathbf{x}_\varepsilon] \lessdot \mathbb{P}_\beta$ for $\varepsilon < \delta$; hence $\mathbb{R}_{\beta,\gamma}[\mathbb{P}_\beta, \bar{\mathbf{x}}]$ is well defined for $\gamma \in [\beta, \lambda^+]$.

[Why? Again we shall use 1.1(3).

First, $\mathbb{P}_\beta[\mathbf{x}_\varepsilon] = \bigcup\{\mathbb{P}_{\beta_j}[\mathbf{x}_\varepsilon]\colon\ j < i\}$ but $j < i \Rightarrow \mathbb{P}_{\beta_j}[\mathbf{x}_\varepsilon] \subseteq \mathbb{P}_{\beta_j} \subseteq \mathbb{P}_\beta$ so clearly $\mathbb{P}_\beta[\mathbf{x}_\varepsilon] \subseteq \mathbb{P}_\beta$.

Second, $\mathbb{P}_\beta[\mathbf{x}_\varepsilon] \subseteq_{\mathrm{ic}} \mathbb{P}_\beta$, because if $p, q \in \mathbb{P}_\beta[\mathbf{x}_\varepsilon]$ are incompatible in $\mathbb{P}_\beta[\mathbf{x}_\varepsilon]$ then for some $j < i$ we have $p, q \in \mathbb{P}_{\beta_j}[\mathbf{x}_\varepsilon]$ hence p, q are incompatible in $\mathbb{P}_{\beta_j}[\mathbf{x}_\varepsilon]$, so as $\mathbb{P}_{\beta_j}[\mathbf{x}_\varepsilon] \subseteq_{\mathrm{ic}} \mathbb{P}_{\beta_j}$ they are incompatible in $\mathbb{P}_{\beta_j}$, but $\mathbb{P}_{\beta_j} \lessdot \mathbb{P}_\beta$ so they are incompatible in $\mathbb{P}_\beta$ as required.

Third, if $q \in \mathbb{P}_\beta$ then for some $\alpha(0) < \beta$ we have $q \in \mathbb{P}_{\alpha(0)}$ and so there is $p \in \mathbb{P}_{\alpha(0)}[\mathbf{x}_\varepsilon]$ such that $p \leqslant p' \in \mathbb{P}_{\alpha(0)}[\mathbf{x}_\varepsilon] \Rightarrow p', q$ are compatible in $\mathbb{P}_{\alpha(0)}$. So it suffices to prove $p \leqslant p' \in \mathbb{P}_\beta[\mathbf{x}_\varepsilon] \Rightarrow p', q$ are compatible in $\mathbb{P}_\beta$, so fix such p'. As β is a limit ordinal, $\mathbb{P}_\beta = \bigcup\{\mathbb{P}_\alpha\colon\ \alpha < \beta\}$ hence there is $\alpha(1)$ such that $\alpha(0) \leqslant \alpha(1) < \beta$ and $p' \in \mathbb{P}_{\alpha(1)}[\mathbf{x}_\varepsilon]$. Now $p'' := p' \restriction \alpha(0)$ is well defined and belong to $\mathbb{P}_{\alpha(0)}[\mathbf{x}_\varepsilon]$ and is above p, so by the choice of p there is a common upper bound $q^+ \in \mathbb{P}_{\alpha(0)}$ of q and p''. As $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha\colon\ \alpha < \beta\rangle$ is FS iteration, $q^+ \in \mathbb{P}_{\alpha(0)}$, $p' \in \mathbb{P}_{\alpha(1)}[\mathbf{x}_\varepsilon] \lessdot \mathbb{P}_{\alpha(1)}$ and $p' \restriction \alpha(0) \leqslant q^+$, clearly there is a common upper bound $r \in \mathbb{P}_{\alpha(1)} \lessdot \mathbb{P}_\beta$ of p', q^+ so r exemplifies p', q are compatible in $\mathbb{P}_\beta$. So we have finished proving $(*)_2$.]

Let $\underset{\sim}{D}'_{\beta,t} = \bigcup\{\underset{\sim}{D}_{\alpha,t}\colon\ \alpha = \beta_j$ for some $j < i$ so $\alpha < \beta\}$. Clearly $s \leqslant_{\mathscr{T}} t \Rightarrow \underset{\sim}{D}'_{\beta,s} \subseteq \underset{\sim}{D}'_{\beta,t}$ so the main point is to prove not just $\Vdash_{\mathbb{P}_\beta}$ "$\emptyset \notin \mathrm{fil}(\underset{\sim}{D}'_{\beta,t})$", but that moreover $\gamma \in [\beta, \lambda^+] \wedge \gamma \in (S \cap E) \cup \{\lambda^+\} \Rightarrow \Vdash_{\mathbb{R}_{\beta,\gamma}[\mathbb{P}_\beta, \bar{\mathbf{x}}]}$ "$\emptyset \notin \mathrm{fil}(D'_{\beta,\gamma,t})$" where $D'_{\beta,\gamma,t} = \bigcup\{\underset{\sim}{D}_{\gamma,t}[\mathbf{x}_\varepsilon]\colon\ \varepsilon < \delta\} \cup D'_{\alpha,t} = \bigcup\{\underset{\sim}{D}_{\gamma,t}[\mathbf{x}_\varepsilon]\colon\ \varepsilon < \delta\} \cup \bigcup\{\underset{\sim}{D}_{\alpha,t}\colon\ \alpha = \beta_j$ for some $j < i\}$. Fixing such γ, again as $\langle \underset{\sim}{D}^{\mathbf{x}_\varepsilon}_{\gamma,t}\colon\ \varepsilon < \delta\rangle$ is increasing and $\langle \underset{\sim}{D}_{\alpha,t}\colon\ \alpha = \beta_j$ for some $j < i\rangle$ is increasing, it suffice to prove $\Vdash_{\mathbb{R}_{\beta,\gamma}[\mathbb{P}_\beta, \bar{\mathbf{x}}]}$ "$\emptyset \in \mathrm{fil}(\underset{\sim}{D}^{\mathbf{x}_\varepsilon}_{\gamma,t} \cup \underset{\sim}{D}_{\alpha,t})$", for any $\varepsilon < \delta$ and $\alpha = \beta_j$, $j < i$. For this it suffices to prove:

$(*)_3$ if (A) then (B) where
- (A)
  - (a) $p = (p_1, p_2) \in \mathbb{R}_{\beta,\gamma}[\mathbb{P}_\beta, \bar{\mathbf{x}}]$
  - (b) $t \in \mathscr{T}$
  - (c) $\alpha = \beta_j < \beta$ and $\underset{\sim}{A} \in \underset{\sim}{D}_{\alpha,t}$ a $\mathbb{P}_\alpha$-name of a subset of $\mathbb{N}$
  - (d) $\varepsilon < \delta$ and $\underset{\sim}{B} \in \underset{\sim}{D}^{\bar{\mathbf{x}}_\varepsilon}_{\gamma,t}$ a $\mathbb{P}^{\mathbf{x}_\varepsilon}_\gamma$-name of a subset of $\mathbb{N}$
  - (e) $n_* \in \mathbb{N}$
- (B) $p \Vdash_{\mathbb{R}_{\beta,\gamma}[\mathbb{P}_\beta, \bar{\mathbf{x}}]}$ "$\underset{\sim}{A} \cap \underset{\sim}{B} \nsubseteq [0, n_*)$".

Proof of $(*)_3$. Let (ε_0, p_0) be a witness for $(p_1, p_2) \in \mathbb{R}_{\beta,\gamma}[\mathbb{P}, \bar{\mathbf{x}}]$; as we can increase ε_0, by $\boxplus_3$, and we can increase ε, without loss of generality $\varepsilon_0 = \varepsilon$.

Without loss of generality $p_0, p_2 \in \mathbb{P}_\alpha$, as we can increase α, moreover as $\iota < i \Rightarrow \beta_{\iota+1} \in E \cap S$, similarly without loss of generality $\alpha \in S \cap E$. Let $p_2^* \in \mathbb{P}_\alpha$ be a common upper bound of p_0, p_2. We define a $\mathbb{P}^{\mathbf{x}_\varepsilon}_\alpha$-name $\underset{\sim}{A}'$ by:

$(*)_{3.1}$ if $\mathbf{G} \subseteq \mathbb{P}^{\mathbf{x}_\varepsilon}_\alpha$ is generic over $\mathbf{V}$ then $\underset{\sim}{A}'[\mathbf{G}] = \{n\colon$ some $q \in \mathbb{P}_\alpha/\mathbf{G}$ forces $n \in \underset{\sim}{A}$ and if $p_2^* \in \mathbb{P}_\alpha/\mathbf{G}$ then $\mathbb{P}_\alpha \models$ "$p_2^* \leqslant q$"$\}$.

Easily

$(*)_{3.2}$ $\underset{\sim}{A}'$ is a $\mathbb{P}^{\mathbf{x}_\varepsilon}_\alpha$-name of a subset of $\mathbb{N}$

$(*)_{3.3}$ $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{A} \subseteq \underset{\sim}{A}'$".

As $\mathbf{x}_\varepsilon \in \mathbf{Q}$ and $\alpha \in S \cap E \subseteq S \cap E_{\mathbf{x}_\varepsilon}$ and $\mathbb{P}^{\mathbf{x}_\varepsilon}_\alpha \lessdot \mathbb{P}_\alpha$ and $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{D}^{\mathbf{x}_\varepsilon}_{\alpha,t} \subseteq \underset{\sim}{D}_{\alpha,t}$", it follows that

$(*)_{3.4}$ $\Vdash_{\mathbb{P}_\alpha[\mathbf{x}_\varepsilon]}$ "$\underset{\sim}{A}' \in \underset{\sim}{D}^{\mathbf{x}_\varepsilon}_{\alpha,t}$".

But $\mathbb{P}_\alpha[\mathbf{x}_\varepsilon] \lessdot \mathbb{P}_\gamma[\mathbf{x}_\varepsilon]$ hence, recalling (A)(d) of $(*)_3$:

$(*)_{3.5}$ $\Vdash_{\mathbb{P}_\gamma[\mathbf{x}_\varepsilon]}$ "$\underset{\sim}{A}' \in \underset{\sim}{D}_{\gamma,t}[\mathbf{x}_\varepsilon]$ and $\underset{\sim}{B} \in \underset{\sim}{D}_{\gamma,t}[\mathbf{x}_\varepsilon]$"

hence

$(*)_{3.6}$ $\Vdash_{\mathbb{P}_\gamma[\mathbf{x}_\varepsilon]}$ "$\underset{\sim}{A}' \cap \underset{\sim}{B} \in \underset{\sim}{D}_{\gamma,t}[\mathbf{x}_\varepsilon]$".

Let $p_0' \in \mathbb{P}_\beta[\mathbf{x}_\varepsilon]$ be such that $p_0' \leqslant p'' \in \mathbb{P}_\beta[\mathbf{x}_\varepsilon] \Rightarrow p'', p_2^*$ are compatible and without loss of generality $p_0 \leqslant p_0'$. Let $p_3 \in \mathbb{P}_\gamma[\mathbf{x}_\varepsilon]$ be above p_1 and p_0'; by $(*)_{3.6}$ there are q_1 and n such that: $p_3 \leqslant q_1 \in \mathbb{P}_\gamma[\mathbf{x}_\varepsilon]$ and $n \geqslant n_*$ and $q_1 \Vdash_{\mathbb{P}_\gamma[\mathbf{x}_\varepsilon]}$ "$n \in \underset{\sim}{A}' \cap \underset{\sim}{B}$".

Let $q_0 = q_1 \restriction \beta$, it belongs to $\mathbb{P}_\beta[\mathbf{x}_\varepsilon]$; clearly $q_0 \leqslant q \in \mathbb{P}_\beta[\mathbf{x}_\varepsilon] \Rightarrow p_2^*, q$ are compatible in $\mathbb{P}_\gamma[\mathbf{x}_\varepsilon]$. Also clearly $p'_0 \leqslant q_0 \in \mathbb{P}_\alpha[\mathbf{x}_\varepsilon]$ so there is r_1 such that $q_0 \leqslant r_1 \in \mathbb{P}_\alpha[\mathbf{x}_\varepsilon]$ and r_1 forces a truth value to "$n \in \underset{\sim}{A}'$" so as r_1 is compatible with q_1, necessarily $r_1 \Vdash$ "$n \in \underset{\sim}{A}'$". So $p_0 \leqslant q_0 \leqslant r_1 \in \mathbb{P}_\beta[\mathbf{x}_\varepsilon]$.

By the definition of $\underset{\sim}{A}'$ and the choice of p_0, there is $q_2 \in \mathbb{P}_\gamma[\mathbf{x}_\varepsilon]$ such that:

$(*)_{3.7}$ (a) $\mathbb{P}_\alpha \models$ "$p_2^* \leqslant q_2$ and $q_0 \leqslant r_1 \leqslant q_2$"
(b) $q_2 \Vdash_{\mathbb{P}_\alpha}$ "$n \in \underset{\sim}{A}$".

Let $\alpha(1) < \beta$ be $\geqslant \alpha$ such that $q_1 \restriction \beta \in \mathbb{P}_{\alpha(1)}[\mathbf{x}_\varepsilon]$; as $(q_1 \restriction \beta)\restriction \alpha = q_0 \leqslant r_1 \leqslant q_2$ and as $\langle \mathbb{P}_\gamma, \underset{\sim}{\mathbb{Q}}_\gamma \colon \gamma < \beta\rangle$ is an FS iteration, clearly $q_1 \restriction \beta, q_2$ are compatible in $\mathbb{P}_{\alpha(1)}$ and let $q_4 \in \mathbb{P}_{\alpha(1)}$ be a common upper bound of $(q_1 \restriction \beta), q_2$. Let $q'_0 \in \mathbb{P}_{\alpha(1)}[\mathbf{x}_\varepsilon]$ be such that $q'_0 \leqslant q \in \mathbb{P}_{\alpha(1)}[\mathbf{x}_\varepsilon] \Rightarrow q, q_4$ are compatible in $\mathbb{P}_{\alpha(1)}$, so as $(q_1 \restriction \beta) \leqslant q_4$, without loss of generality $(q_1 \restriction \beta) \leqslant q'_0$.

$(*)_{3.8}$ $q'_0 \in \mathbb{P}_\beta[\mathbf{x}_\varepsilon]$ and (ε, q'_0) witness $(q_1, q_4) \in \mathbb{R}_{\beta,\gamma}[\mathbb{P}_\beta, \bar{\mathbf{x}}]$.

[Why? As $\mathbf{x}_\varepsilon \in \mathbf{Q}$ and $q_1 \restriction \beta = q_0 \leqslant q'_0$ clearly $q'_0 \Vdash_{\mathbb{P}_\beta[\mathbf{x}_\varepsilon]}$ "$q_1 \in \mathbb{P}_\gamma[\mathbf{x}_\varepsilon]/\underset{\sim}{G}_{\mathbb{P}_\beta[\mathbf{x}_\varepsilon]}$".

For proving $q'_0 \Vdash_{\mathbb{P}_\beta[\mathbf{x}_\varepsilon]}$ "$q_4 \in \mathbb{P}_\beta/\underset{\sim}{G}_{\mathbb{P}_\beta[\mathbf{x}_\varepsilon]}$" recall the choice of q'_0.]

$(*)_{3.9}$ $(q_1, q_4) \Vdash_{\mathbb{R}_{\beta,\gamma}[\mathbb{P}_\beta, \bar{\mathbf{x}}]}$ "$n \in \underset{\sim}{A} \cap \underset{\sim}{B} \backslash [0, n_*)$".

[Why? First, $q_1 \Vdash_{\mathbb{P}_\gamma[\mathbf{x}_\varepsilon]}$ "$n \in \underset{\sim}{B}$" by the choice of q_1 hence $(q_1, q_4) \Vdash_{\mathbb{R}_{\beta,\gamma}[\mathbb{P}_\beta, \bar{\mathbf{x}}]}$ "$n \in \underset{\sim}{B}$" recalling $\mathbb{P}_\gamma[\mathbf{x}_\varepsilon] \lessdot \mathbb{R}_{\beta,\gamma}[\mathbb{P}_\beta, \bar{\mathbf{x}}]$ by $\boxplus_7$.

Second, $q_4 \Vdash_{\mathbb{P}_\beta}$ "$n \in \underset{\sim}{A}$" because $q_2 \Vdash_{\mathbb{P}_\alpha}$ "$n \in \underset{\sim}{A}$" and $q_2 \leqslant q_4, \mathbb{P}_\alpha \lessdot \mathbb{P}_\beta$ and so $(q_1, q_4) \Vdash_{\mathbb{R}_{\beta,\gamma}[\mathbb{P}_\beta, \bar{\mathbf{x}}]}$ "$n \in \underset{\sim}{A}$" because $\mathbb{P}_\beta \lessdot \mathbb{R}_{\beta,\gamma}[\mathbb{P}_\beta, \bar{\mathbf{x}}]$ by $\boxplus_6$.

Third, $n \geqslant n_*$ recalling the choice of n. So $(*)_{3.9}$ holds.]

Together we have proved $(*)_3$.

Lastly, clearly $\beta_i \in E$ and let $\bar{\underset{\sim}{D}}_\beta = \bar{\underset{\sim}{D}}'_\beta$. If $\beta = \beta_i \notin S$ we are done. So assume $\beta \in S$; by the induction hypothesis $\alpha = \beta_j < \beta \Rightarrow \Vdash_{\mathbb{P}_{\beta_{j+1}}}$ "$\bar{\underset{\sim}{D}}_{\beta_{j+1}}$ is ultra $\mathscr{T}$-filter system", and $\bar{\underset{\sim}{D}}_\alpha$ increases with α, also necessarily $\mathrm{cf}(\beta) = \lambda$ hence $\Vdash_{\mathbb{P}_\beta}$ "$\langle \bigcup\{\underset{\sim}{D}_{\alpha,t} \colon \alpha < \beta\} \colon t \in \mathscr{T}\rangle$ is ultra hence $\bar{\underset{\sim}{D}}'$ is ultra so we are done.

Case 3: $i = j+1, \beta_j \notin S \cup S_0$.

Let $\gamma \in (\beta_j, \lambda^+]$ and $\mathbb{R} = \mathbb{R}_{\beta_j,\gamma}[\mathbb{P}_{\beta_j}, \bar{\mathbf{x}}]$, recalling $\boxplus_5$ we know $\mathbb{R}$ satisfies the c.c.c., by $\boxplus_6$ we know $\mathbb{P}_{\beta_j} \lessdot \mathbb{R}$ and by $\boxplus_7$ we know $\varepsilon < \delta \Rightarrow \mathbb{P}_\gamma[\mathbf{x}_\varepsilon] \lessdot \mathbb{R}$. For $t \in \mathscr{T}$, let $\underset{\sim}{D}'_{\beta_j,\gamma,t} = \bigcup\{\underset{\sim}{D}_{\gamma,t}[\mathbf{x}_\varepsilon] \colon \varepsilon < \delta\} \cup \underset{\sim}{D}_{\beta_j,t}$, noting $\underset{\sim}{D}_{\beta_j,t} = \bigcup\{\underset{\sim}{D}_{\beta_\iota,t} \colon \iota \leqslant j\}$, so by the induction hypothesis, $\Vdash_{\mathbb{R}}$ "$\emptyset \notin \mathrm{fil}(\underset{\sim}{D}'_{\beta_j,\gamma,t})$" so $\bar{\underset{\sim}{D}}'_{\beta_j,\gamma,t} = \langle \underset{\sim}{D}'_{\beta_j,\gamma,t} \colon t \in \mathscr{T}\rangle$ is a $\mathbb{R}_{\beta_j,\gamma}[\mathbb{P}_{\beta_j}]$-name of a $\mathscr{T}$-filter system. Hence there is $\bar{\underset{\sim}{D}}''_{\beta_i,\gamma} = \langle D''_{\beta_i,\gamma,t} \colon t \in \mathscr{T}\rangle$, a $\mathbb{R}$-name of an ultra $\mathscr{T}$-filter system above $\bar{D}'_{\beta_i,\gamma}$, without loss of generality $D''_{\beta_i,\gamma,t} = \mathrm{fil}(D''_{\beta_i,\gamma,t})$ for $t \in \mathscr{T}$. In particular this holds for $\gamma = \lambda^+$ hence E_i^* is a club of λ^+ where

$(*)_4$ $E_i^* = \{\gamma < \lambda^+ \colon \gamma$ is a limit ordinal from E and if $\xi < \gamma$ then $\langle D''_{\beta_i,\lambda^+,t} \cap \mathscr{P}(\mathbb{N})^{\mathbf{V}[\mathbb{P}_\xi]} \colon t \in \mathscr{T}\rangle$ is a $\mathbb{R}_{\beta_j,\xi_1}[\mathbb{P}_{\beta_j}, \bar{\mathbf{x}}]$-name for some $\xi_1 < \gamma\}$.

So we can choose $\beta_i = \beta(i) \in E_i^* \cap E \cap S \backslash (\beta_j + 1)$.

Let $\mathbb{P}_{\beta_i} = \mathbb{R}_{\beta_j,\beta_i}[\mathbb{P}_{\beta_j}, \bar{\mathbf{x}}]$ and similarly $\mathbb{P}_\alpha = \mathbb{R}_{\beta_j,\alpha}[\mathbb{P}_{\beta_j}, \bar{\mathbf{x}}]$ for $\alpha \in (\beta_j, \beta_i)$ and $\bar{\underset{\sim}{D}}_{\beta_i} = \langle D''_{\beta_i,\lambda^+,t} \cap \mathscr{P}(\mathbb{N})^{\mathbf{V}[\mathbb{P}_{\beta(i)}]} \colon t \in \mathscr{T}\rangle$.

Also the choice of $\underset{\sim}{\mathbb{Q}}_\alpha, \underset{\sim}{g}_\alpha$ for $\alpha \in [\beta_j, \beta_i)$ is dictated by clause (g) of $\boxdot$ hence also of f_α and it is easy to check that all the clauses in the induction hypothesis are satisfied.

Case 4: $i = j+1, \beta_j \in S$.

So $\Vdash_{\mathbb{P}_{\beta_j}}$ "$\bar{\underset{\sim}{D}}_{\beta_j}$ is an ultra $\mathbb{P}_{\beta_j}$-filter system". Let $\beta = \beta_j$.

Let $\underset{\sim}{\mathbb{Q}}_\beta = \mathbb{Q}_{\underset{\sim}{D}_\beta}, \mathbb{P}_{\beta+1} = \mathbb{P}_\beta * \mathbb{Q}_{\underset{\sim}{D}_\beta}$. By Claim 1.9, $\mathbb{P}_{\beta+1}[\mathbf{x}_\varepsilon] = \mathbb{P}_\beta[\mathbf{x}_\varepsilon] * \underset{\sim}{\mathbb{Q}}_{\underset{\sim}{D}_\beta[\mathbf{x}_\varepsilon]} \lessdot \mathbb{P}_\beta * \mathbb{Q}_{\underset{\sim}{D}_\beta} = \mathbb{P}_{\beta_j+1}$ for $\varepsilon < \delta$. So $\mathbb{R}_{\beta+1,\gamma}[\mathbb{P}_{\beta+1}, \bar{\mathbf{x}}]$ is well defined for $\gamma \in [\beta+1, \lambda^+]$.

For $t \in \mathscr{T}$ let $\underset{\sim}{D}'_{\beta+1,s}$ be the dual of $\underset{\sim}{\mathrm{id}}_{\underset{\sim}{\mathbf{d}}_{t(\beta),s}}[\mathbb{P}_\beta]$, a $\mathbb{P}_{\beta+1}$-name.

$(*)_5$ $\Vdash_{\mathbb{R}_{\beta+1,\gamma}[\mathbb{P}_{\beta+1},\bar{\mathbf{x}}]}$ "$\emptyset \notin \mathrm{fil}(\bigcup\{\underset{\sim}{D}_{\gamma,s}[\mathbf{x}_\varepsilon] \colon \varepsilon < \delta\} \cup \underset{\sim}{D}'_{\beta,s})$" for $\gamma \in (\beta, \lambda^+]$.

Note that for (β, γ) we know the parallel statements.

$(*)_6$ convention: we write $(p_1, p_2, \underset{\sim}{p}_3) = (p_1, (p_2, \underset{\sim}{p}_3))$ for members of $\mathbb{R}_{\beta+1,\gamma}[\mathbb{P}_{\beta+1}, \bar{\mathbf{x}}]$, where we treat $\mathbb{P}_{\beta+1}$ as $\mathbb{P}_\beta * \underset{\sim}{\mathbb{Q}}_{\bar{D}_\beta}$, so $p_2 \in \mathbb{P}_\beta$ and $\Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{p}_3 \in \underset{\sim}{\mathbb{Q}}_{\bar{D}_\beta}$" and $\mathrm{tr}(\underset{\sim}{p}_3)$ is an object not just a name.

We need

$(*)_7$ if (A) then (B) where

(A) (a) $p = (p_1, p_2, \underset{\sim}{p}_3) \in \mathbb{R}_{\beta+1,\gamma}[\mathbb{P}_{\beta+1}, \bar{\mathbf{x}}]$
(b) $t \in \mathscr{T}$
(c) $\underset{\sim}{A}$ is a $\mathbb{P}_{\beta+1}$-name of a member of $\underset{\sim}{D}'_{\beta,t}$ that is, $\Vdash_{\mathbb{P}_{\beta+1}}$ "$\mathbb{N} \backslash \underset{\sim}{A}$ is $\underset{\sim}{\mathbf{d}}_{t(\beta),t}$-null"
(d) $\varepsilon < \delta$ and $\underset{\sim}{B}$ is a $\mathbb{P}_\gamma[\mathbf{x}_\varepsilon]$-name of a member of $\underset{\sim}{D}_{\gamma,t}[\mathbf{x}_\varepsilon]$
(e) $n_* \in \mathbb{N}$

(B) $p \Vdash_{\mathbb{R}_{\beta+1,\gamma}[\mathbb{P}_{\beta+1},\bar{\mathbf{x}}]}$ "$\underset{\sim}{A} \cap \underset{\sim}{B} \nsubseteq [0, n_*)$".

First note

$(*)_{7.1}$ (a) let $(\varepsilon, (p_0, \underset{\sim}{p}'_3))$ where $(p_0, \underset{\sim}{p}'_3) \in \mathbb{P}_\beta[\mathbf{x}_\varepsilon] * \mathbb{Q}_{\underset{\sim}{\bar{D}}_\beta[\mathbf{x}_\varepsilon]}$ witness $p \in \mathbb{R}_{\beta+1,\gamma}[\mathbb{P}_{\beta+1}, \bar{\mathbf{x}}]$
(b) let $q_2 \in \mathbb{P}_\beta$ be above p_0, p_2
(c) let $q_0 \in \mathbb{P}_\beta$ be such that $q_0 \leqslant q' \in \mathbb{P}_\beta[\mathbf{x}_\varepsilon] \Rightarrow q_2, q'$ are compatible
(d) let $\underset{\sim}{B}'$ be the following $\mathbb{P}_{\beta+1}[\mathbf{x}_{\varepsilon+1}]$-name $\{n\colon$ there is $q \in \mathbb{P}_\gamma[\mathbf{x}_\varepsilon]/\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\beta+1}[\mathbf{x}_\varepsilon]}$ forcing $n \in \underset{\sim}{B}$ above p_1 when $p_1 \in \mathbb{P}_\gamma[\mathbf{x}_\varepsilon]/\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\beta+1}[\mathbf{x}_\varepsilon]}\}$.

Next consider

$(*)_{7.2}$ $\Vdash_{\mathbb{P}_{\beta+1}}$ "$\underset{\sim}{A} \cap \underset{\sim}{B}' \nsubseteq [0, n_*)$".

Why is $(*)_{7.2}$ true? Note that $\Vdash_{\mathbb{P}_{\beta+1}[\mathbf{x}_\varepsilon]}$ "$\underset{\sim}{B}' \in (\underset{\sim}{\mathrm{id}}_{\mathbf{d}(t_\beta,s)})^+$" by clause (g) of Definition 2.5, as $\Vdash_{\mathbb{P}_\gamma[\mathbf{x}_\varepsilon]}$ "$\underset{\sim}{B} \in \underset{\sim}{D}_{\gamma,s}$ and $\underset{\sim}{B}' \subseteq \underset{\sim}{B}$". Now apply Claim 1.14 for $\mathbb{P}_{\beta+1}[\mathbf{x}_\varepsilon] = \mathbb{P}_\beta[\mathbf{x}_\varepsilon] * \mathbb{Q}_{\underset{\sim}{D}_{\beta,t(\alpha)}[\mathbf{x}_\varepsilon]}$ and $\mathbb{P}_{\beta+1} = \mathbb{P}_\beta * \mathbb{Q}_{\underset{\sim}{D}_{\beta,t(\alpha)}}$.

Why is $(*)_{7.2}$ enough for proving $(*)_7$? As in the proof of Case 2, only much easier.

Case 5: $i = j+1$, $\beta_j \in S_0$.

Let $\beta = \beta_j$; and let $\mathbb{P}_{\beta+1} = \mathbb{P}_\beta * \mathbb{Q}_{\mathrm{fil}(\emptyset)}$ so $\underset{\sim}{\mathbb{Q}}_\beta = \underset{\sim}{\mathbb{Q}}_{\mathrm{fil}(\emptyset)}$, and again $\mathbb{P}_\beta[\mathbf{x}_\varepsilon] * \mathbb{Q}_{\mathrm{fil}(\emptyset)} \lessdot \mathbb{P}_\beta * \mathbb{Q}_{\mathrm{fil}(\emptyset)}$ by 1.9. Clearly $\mathbb{R}_{\beta+1,\gamma}[\mathbb{P}_{\beta+1}, \bar{\mathbf{x}}]$ is well defined for $\gamma \in [\beta+1, \lambda^+]$. We let $\underset{\sim}{D}'_{\beta+1,t} = \bigcup\{\underset{\sim}{D}_{\alpha,t}\colon \alpha \in S \cap E\} \cup \{\underset{\sim}{u}_{\beta,s,n}\colon s \leqslant_{\mathscr{T}} t$ and $n \in \mathbb{N}\}$, a $\mathbb{P}_{\beta+1}$-name.

We have to prove the parallel of $(*)_5$, i.e.

$(*)_8$ $\Vdash_{\mathbb{R}_{\beta+1,\gamma}[\mathbb{P}_{\beta+1},\bar{\mathbf{x}}]}$ "$\emptyset \notin \mathrm{fil}(\underset{\sim}{D}'_{\alpha,t})$" for $\gamma \in [\beta+1, \lambda^+]$ and $t \in \mathscr{T}$.

By 2.9 it suffices to prove

$(*)_9$ $\Vdash_{\mathbb{R}_{\beta+1,\gamma}[\mathbb{P}_{\beta+1},\bar{\mathbf{x}}]}$ "$\emptyset \notin \mathrm{fil}(\{\underset{\sim}{u}_{\beta,s}\colon s \leqslant_{\mathscr{T}} t\})$" for $t \in \mathscr{T}$.

Now it is like Case 4 only easier. $\square_{2.11}$

Claim 2.13. *If $\mathbf{x} \in \mathbf{Q}$ and $\theta \in \Theta_2$ <u>then</u> we can find a pair $(\mathbf{y}, \mathbf{j}_*)$ such that*

(a) $\mathbf{x} \leqslant_{\mathbf{Q}} \mathbf{y}$
(b) $\mathbf{j}_*$ *is an isomorphism from* $(\mathbb{P}^{\mathbf{x}})^\theta/E_\theta$ *onto* $\mathbb{P}^{\mathbf{y}}$ *extending* $\mathbf{j}_{**}^{-1}$ *where* $\mathbf{j}_{**}$ *is the canonical embedding of* $\mathbb{P}^{\mathbf{x}}_\kappa$ *into* $(\mathbb{P}^{\mathbf{x}}_\kappa)^\theta/E_\theta$
(c) $\mathbf{j}_*$ *maps* $(\mathbb{P}^{\mathbf{x}}_\alpha)^\theta/E_\theta$ *onto* $\mathbb{P}^{\mathbf{y}}_\alpha$ *for any* $\alpha < \lambda^+$ *satisfying* $\mathrm{cf}(\alpha) \neq \theta$
(d) *note that* $\mathbf{j}_*$ *maps* $\mathbf{j}_{**}(\mathbb{P}^{\mathbf{x}}_\alpha)$ *to a* $\lessdot$*-subforcing of* $\mathbb{P}^{\mathbf{y}}_\alpha$ *for* $\alpha \leqslant \lambda^+$ *satisfying* $\mathrm{cf}(\alpha) \neq \theta$.

Before proving 2.13 recall:

Definition 2.14. 1) For a c.c.c. forcing notion $\mathbb{P}$ and $\mathbb{P}$-name $\underset{\sim}{A}$ of a subset of $\mathbb{N}$ we say that $\mathbf{p} = \langle (p_{n,m}, \mathbf{t}_{n,m})\colon m, n < \omega\rangle$ represents $\underset{\sim}{A}$ <u>when</u>:

(a) $p_{n,m} \in \mathbb{P}$ and $\mathbf{t}_{n,m}$ is a truth value
(b) for each n, $\langle p_{n,m}\colon m < \omega\rangle$ is a maximal antichain of $\mathbb{P}$
(c) for $n, m < \omega$ we have $p_{n,m} \Vdash_{\mathbb{P}}$ "$n \in \underset{\sim}{A}$ <u>iff</u> $\mathbf{t}_{n,m}$".

2) For $\mathbf{p}$ as in part (1) let $\underset{\sim}{A}_{\mathbf{p}}$ be the canonical $\mathbb{P}$-name represented by $\mathbf{p}$.

Fact 2.15. 1) *If $\mathbb{P}$ is a c.c.c. forcing notion and $\underset{\sim}{A}$ is a $\mathbb{P}$-name of a subset of $\mathbb{N}$ <u>then</u> some $\langle (p_{n,m}, \mathbf{t}_{n,m})\colon n,m<\omega\rangle$ represents $\underset{\sim}{A}$.*

2) *If $\mathbb{P}$ is a c.c.c. forcing notion and $\underset{\sim}{A}'$, $\underset{\sim}{A}''$ are $\mathbb{P}$-names of subsets of ω, both represented by $\langle (p_{n,m}, \mathbf{t}_{n,m})\colon n,m<\omega\rangle$ <u>then</u> $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{A}' = \underset{\sim}{A}''$".*

3) *For a sequence $\bar{\mathbf{t}} = \langle \mathbf{t}_{n,m}\colon n,m<\omega\rangle$ of truth values, for some formula $\varphi = \varphi^0_{\bar{\mathbf{t}}}(\bar{x}) \in \mathbb{L}_{\aleph_1,\aleph_1}(\tau)$, $\tau = \{\leqslant\}$ where $\bar{x} = \langle x_{n,m}\colon n<\omega\rangle$ we have: for every c.c.c. forcing notion $\mathbb{P}$ and $p_{n,m}\in\mathbb{P}$ $(n,m<\omega)$ we have:*

⊛ *$\mathbb{P} \models$ "$\varphi(\langle p_{n,m}\colon n,m<\omega\rangle)$ iff $\langle (p_{n,m}, \mathbf{t}_{n,m})\colon n,m<\omega\rangle$ represents a $\mathbb{P}$-name of a non-empty subset of ω".*

4) *For $k<\omega$, sequences $\bar{\mathbf{t}}^\ell = \langle \mathbf{t}^\ell_{n,m}\colon n,m<\omega\rangle$ of truth values for $\ell\leqslant k$ for some $\mathbb{L}_{\aleph_1,\aleph_1}(\tau)$-formula $\varphi = \varphi^k_{\bar{\mathbf{t}}^0,\dots,\bar{\mathbf{t}}^k}(y,\bar{x}^0,\dots,\bar{x}^k)$ where $\bar{x}^\ell = \langle x^\ell_{n,m}\colon n,m<\omega\rangle$ we have:*

⊛ *for every $q, p^\ell_{n,m}\in\mathbb{P}$ $(n,m<\omega,\ \ell\leqslant k)$, $\mathbb{P}$ a c.c.c. forcing notion we have: $\mathbb{P}\models\varphi[q,\langle p^0_{n,m}\colon n,m<\omega\rangle,\langle p^1_{n,m}\colon n,m<\omega\rangle,\dots,\langle p^k_{n,m}\colon n,m<\omega\rangle]$ <u>iff</u> $\langle (p^\ell_{n,m},\mathbf{t}^\ell_{n,m})\colon n,m<\omega\rangle$ represents a $\mathbb{P}$-name of a subset of ω which we call $\underset{\sim}{A}_\ell$, for $\ell\leqslant k$ and $q\Vdash_{\mathbb{P}}$ "$\underset{\sim}{A}_k$ and $\mathbb{N}\backslash\underset{\sim}{A}_k$ do not almost include $\underset{\sim}{A}_0\cap\underset{\sim}{A}_1\cap\dots\cap\underset{\sim}{A}_{k-1}$".*

Proof. Easy. $\square_{2.15}$

Remark 2.16. In 2.15 we can treat any other relevant properties of such $\mathbb{P}$-names.

Proof of 2.13. Let χ be large enough, $\mathbf{x}\in\mathscr{H}(\chi)$ and $\mathfrak{B} = (\mathscr{H}(\chi),\in)^\theta/E_\theta$ and let $\mathbf{j}$ the canonical embedding of $(\mathscr{H}(\chi),\in)$ into $\mathfrak{B}$.

We now define

⊞ (a) $\mathbb{P}_\alpha$ is $(\mathbb{P}^{\mathbf{j}(\mathbf{x})}_{\mathbf{j}(\alpha)})^{\mathfrak{B}}$ if $\alpha\leqslant\lambda^+$, $\mathrm{cf}(\alpha)\neq\theta$ and $\mathbb{P}_\alpha = \bigcup\{\mathbb{P}_\beta\colon \beta<\alpha\}$ if $\alpha<\lambda^+\wedge\mathrm{cf}(\alpha)=\theta$
(b) $I_{<\alpha} = \bigcup\{(I^{\mathbf{j}(\mathbf{x})}_{<\mathbf{j}(\beta+1)})^{\mathfrak{B}}\colon \beta<\alpha\}$ and $E = E^{\mathbf{y}} = E^{\mathbf{x}}$
(c) f_α, a function with domain $\mathbb{P}_\alpha$ is defined by:
 (α) $f_\alpha(p)$ is a function with domain $\{a\colon \mathfrak{B}\models a\in\mathrm{Dom}(\mathbf{j}(f_\alpha(p)))\}$
 (β) $f_\alpha(p)(a) = \mathbf{j}^{-1}((f_\alpha(p)(a))^{\mathfrak{B}})$
(d) $(I_\alpha, \underset{\sim}{g}_\alpha, \mathbb{Q}_\alpha)$ is defined naturally for $\alpha<\lambda^+$:
 (α) if $\mathrm{cf}(\alpha)\neq\theta$ as $(\mathbf{j}(\mathbb{Q}^{\mathbf{x}}_\alpha(p)))^{\mathfrak{B}}$
 (β) if $\mathrm{cf}(\alpha)=\theta$, it is $\{p\in\mathbb{P}_{\alpha+1}\colon \mathrm{dom}(p)\subseteq\bigcap_{\beta<\alpha}[\mathbf{j}(\beta),\mathbf{j}(\alpha+1))^{\mathfrak{B}}\}$, etc.
(e) $E = E_{\mathbf{x}}$.

We like to choose $\mathbb{P}^{\mathbf{y}}_\alpha = \mathbb{P}_\alpha$, a pedantic objection is that $\mathbf{j}$ is not the identity, moreover $\mathbb{P}_\alpha\nsubseteq\mathscr{H}(\lambda^{++})$; so $\mathbb{P}^{\mathbf{x}}_\alpha\nsubseteq\mathbb{P}_\alpha$, by renaming we can overcome this.

Also for $\alpha\in E\cup\{\lambda^+\}$ and $t\in\mathscr{T}$ the $\mathbb{P}^{\mathbf{y}}_\alpha$-name $\underset{\sim}{D}^{\mathbf{y}}_{\alpha,t}$ are naturally defined such that

(∗) $\Vdash_{\mathbb{P}^{\mathbf{y}}_\alpha}$ "$\underset{\sim}{D}^{\mathbf{y}}_{t,\alpha} = \{\underset{\sim}{A}_{\mathbf{p}}\colon$ $\mathbf{p}$ represents some $\mathbb{P}^{\mathbf{y}}_{t,\alpha}$-name of subset of $\mathbb{N}$ and $p\Vdash$ "$\mathbf{p}\in\mathbf{j}(\underset{\sim}{D}^{\mathbf{x}}_{t,\alpha})$" for some $p\in\underset{\sim}{G}_{\mathbb{P}^{\mathbf{y}}_\alpha}\}$.

Almost all the desired properties hold by Łos theorem for $\mathbb{L}_{\aleph_1,\aleph_1}$ as in 2.15. A problem is to show clause $(d)(\alpha)$ of 2.5, being "ultra" which means

⊙ if $\partial\in\Theta_1$, $s\in\mathscr{T}_\partial$, $\alpha\in E\cap S$ then $\Vdash_{\mathbb{P}^{\mathbf{y}}_\alpha}$ "if $A\subseteq\mathbb{N}$ and $A\neq\emptyset$ mod $\underset{\sim}{D}^{\mathbf{y}}_{\alpha,s}$ then for some t we have $s\leqslant_{\mathscr{T}_\partial} t$ and $A\in\underset{\sim}{D}^{\mathbf{y}}_{\alpha,t}$".

Toward this, as $\theta\in\Theta_2$, $\partial\in\Theta_1$ we have $\theta\neq\partial$ hence

⊡ if η be a generic branch of $\mathscr{T}_\partial$ over $\mathbf{V}$ so η is a subset of $\mathscr{T}_\partial$ of order type θ by $<_{\mathscr{T}}$ <u>then</u>
 (α) E_θ is a θ-complete ultrafilter on θ even in $\mathbf{V}[\eta]$;
 (β) $(\mathbb{P}_{\mathbf{x}})^\theta/E_\theta$ is the same in $\mathbf{V}$ and $\mathbf{V}[\eta]$.

[Why? The proof by the division to two cases:

First case: $\theta<\partial$.
The forcing $\mathscr{T}_\partial$ adds to $\mathbf{V}$ no sequence of length $<\partial$ so obvious.

Second case: $\theta>\partial$.
Note that $\mathbf{j}\restriction\mathscr{T}_\partial$ is an isomorphism from $\mathscr{T}_\partial$ onto $(\mathbf{j}\restriction\mathscr{T}_\partial)^{\mathfrak{B}}$ as $|\mathscr{T}_\partial|<\theta$.]

So by $\boxdot$

$\boxtimes$ $\Vdash_{\mathscr{T}_\partial}$ "$\{\underset{\sim}{D}^{\mathbf{y}}_{\alpha,\eta_t}\colon t \in \underset{\sim}{\eta}\}$ is an ultrafilter on $\mathbb{N}$".

This suffices for $\odot$ by 1.5 so we are done. $\square_{2.13}$

We lastly arrive to the desired conclusion.

Conclusion 2.17. *There is $\mathbb{P}$ such that (for our $\mathscr{T}_*$ see 2.1(g), 2.4):*

(a) *$\mathbb{P}$ is a c.c.c. forcing notion of cardinality λ and $\Vdash_{\mathbb{P}}$ "$2^{\aleph_0} = \lambda$"*
(b) *$\mathscr{T}_*$ has cardinality $\Pi\Theta_1 \leqslant \lambda^+$, add no new sequence of length $< \min(\Theta_1)$ of ordinals, collapse no cardinal, change no cofinality*
(c) *$\mathbb{P} \times \mathscr{T}_*$ has cardinality $\leqslant \lambda + \Pi\Theta_1$, collapse no cardinality, change no cofinality and forces $2^{\aleph_0} = \lambda$*
(d) *in $\mathbf{V}^{\mathbb{P}\times\mathscr{T}_*}$ we have $\Theta_1 \subseteq \mathrm{Sp}_\chi$, i.e. for every $\theta \in \Theta_1$ there is a non-principal ultrafilter D of character θ*
(e) *in $\mathbf{V}^{\mathbb{P}\times\mathscr{T}_*}$ we have $\Theta_2 \cap \mathrm{Sp}_\chi = \emptyset$*
(f) *$\mathbb{P} = \mathbb{P}^{\mathbf{x}}_{\delta(*)}$ for some $\mathbf{x} \in \mathbf{Q}$ and $\delta(*) \in E_{\mathbf{x}} \cap S$.*

Remark 2.18. 1) So if $\sup(\Theta_1)$ is strongly inaccessible then $|\mathscr{T}_*| = \sup(\Theta_1)$.
2) Similarly in 3.2 for $\mathbb{Q}$.

Proof. We choose $\mathbf{x}_\varepsilon \in \mathbf{Q}$ by induction on $\varepsilon \leqslant \lambda$ such that

$(*)$ (a) $\mathbf{x}_\varepsilon \in \mathbf{Q}$
(b) $\zeta < \varepsilon \Rightarrow \mathbf{x}_\zeta \leqslant \mathbf{x}_\varepsilon$
(c) if $\varepsilon = \zeta + 1$ and $\mathrm{cf}(\zeta) = \theta \in \Theta_2$ or $\mathrm{cf}(\zeta) \notin \Theta_2 \wedge \theta = \min(\Theta_2)$ then $\mathbf{x}_\varepsilon$ is gotten from $\mathbf{x}_\zeta$ as $\mathbf{y}$ was gotten from $\mathbf{x}$ in 2.13 using E_θ
(d) $\zeta < \varepsilon \Rightarrow E_{\mathbf{x}_\varepsilon} \subseteq E_{\mathbf{x}_\zeta}$.

For $\varepsilon = 0$ use 2.8, for ε successor use 2.13 and for ε limit use 2.11.

Having carried the induction, let $\mathbf{x} = \mathbf{x}_\lambda$. Let $S'_0 = \{\delta \in S_0\colon C^*_\delta \subseteq E_{\mathbf{x}}\}$ so a stationary subset of λ^+. Let $E = \{\delta \in E_{\mathbf{x}}\colon \delta = \sup(\delta \cap S'_0)\}$. Let $\delta(*) \in E$ be such that $\delta(*)$ has cofinality κ. Let $\langle \alpha(\varepsilon)\colon \varepsilon < \kappa\rangle$ be an increasing sequence of members of $E_{\mathbf{x}}$ with limit $\delta(*)$ such that $\varepsilon < \kappa \Rightarrow \alpha(\varepsilon+1) \in S'_0$.

Now letting $\mathbb{P} = \mathbb{P}^{\mathbf{x}}_{\delta(*)}$ recalling $\mathbb{P}^{\mathbf{x}}_{\delta(*)} = \bigcup\{\mathbb{P}^{\mathbf{x}_\varepsilon}_{\delta(*)}\colon \varepsilon < \lambda\}$, it easily satisfies all the requirements but we give some details. We have $\Vdash_{\mathbb{P}}$ "$2^{\aleph_0} \geqslant \lambda$" and $|\mathbb{P}| \geqslant \lambda$ by the choice of $\mathbf{x}_0$ as $\mathbb{P}_1[\mathbf{x}_0] \lessdot \mathbb{P}$, see 2.8; also $\mathbb{P}$ satisfies the c.c.c. (see 2.10(2)) and $\mathbb{P}$ has cardinality $\leqslant \lambda$ (see Definition 2.5, clause (a)) hence $\Vdash_{\mathbb{P}}$ "$2^{\aleph_0} \leqslant \lambda$" recalling $\lambda = \lambda^{\aleph_0}$. So we have shown clause (a) of the conclusion. Clause (b) holds by the choice of $\mathscr{T}_*$ (see end of clause (g) of Hypothesis 2.1). Now $|\mathbb{P}| = \lambda, |\mathscr{T}| \leqslant \Pi\Theta_1$ hence $|\mathbb{P} \times \mathscr{T}_*| \leqslant \lambda + \Pi\Theta_1$ and $\Vdash_{\mathscr{T}_*}$ "$\mathbb{P}$ satisfies the c.c.c." by Hypothesis 2.1(g); hence forcing with $\mathbb{P} \times \mathscr{T}_*$ collapse no cardinal which forcing with $\mathscr{T}_*$ does not collapse; but as $\theta \in \Theta_1 \Rightarrow \theta = \theta^{<\theta}$ and the use of Easton support in the product $\mathscr{T}_*$, forcing with $\mathscr{T}_*$ collapse no cardinal. Similarly forcing with $\mathbb{P} \times \mathscr{T}_*$ changes no cofinality; together clause (c) of 2.17 holds.

As for clause (d), as $\mathscr{T}_*$ is a product, forcing with $\mathscr{T}_*$ adds $\underset{\sim}{\bar{\eta}} = \langle \underset{\sim}{\eta}_\theta\colon \theta \in \Theta_1\rangle, \underset{\sim}{\eta}_\theta$ a θ-branch of $\mathscr{T}_\theta$ so in $\mathbf{V}[\bar{\eta}]$ we have $\bigcup\{\underset{\sim}{D}^{\mathbf{x}_\lambda}_{\delta(*),t}\colon t \in \eta_\theta\}$, which is a $\mathbb{P}$-name $\underset{\sim}{D}_\theta$ of an ultrafilter on $\mathbb{N}$ by 1.5(2), non-principal by 1.2(2). Now for each $t \in \mathscr{T}_\theta$, the filter $\underset{\sim}{D}^{\mathbf{x}_\lambda}_{\delta(*),t}$ is (forced to be) generated by the $\subseteq^*$-decreasing $\langle \underset{\sim}{u}_{\alpha(\varepsilon+1),t,n}\colon \varepsilon < \kappa$ and $n \in \mathbb{N}\rangle$, in the sense that $\underset{\sim}{u}_{\alpha(\varepsilon+1),t,n+1} \subseteq \underset{\sim}{u}_{\alpha(\varepsilon+1),t,n}$ and for $\zeta < \varepsilon$ for some n_* we have $n_1 \in \mathbb{N} \wedge n_2 \in \mathbb{N}\backslash n_* \Rightarrow \underset{\sim}{u}_{\alpha(\varepsilon+1),t,n_2} \subseteq^* u_{\alpha(\zeta+1),t,n_1}$. So $\underset{\sim}{D}_\theta$ is generated by $|\theta| + \kappa = \theta$ sets. Now $\underset{\sim}{\eta}_\theta$ under $<_{\mathscr{T}_\theta}$ has order type θ and no $\underset{\sim}{D}^{\mathbf{x}_\lambda}_{\delta(*),t}$ is an ultrafilter and it increases with t, so clearly $< \theta$ sets do not suffice. Hence $\Vdash_{\mathbb{P}\times\mathscr{T}}$ "$\theta \in \mathrm{Sp}_\chi$ for every $\theta \in \Theta_1$", so clause (d) of 2.17 holds.

Lastly, concerning clause (e), assume that $(p,t) \in (\mathbb{P} \times \mathscr{T}_*)$ forces that "$\underset{\sim}{\mathscr{A}} \subseteq \mathscr{P}(\mathbb{N})$ generates a non-principal ultrafilter $\underset{\sim}{D}$, of character $\theta, \theta = |\underset{\sim}{\mathscr{A}}|$ and $\theta \in \Theta_2$". As $\mathrm{cf}(\lambda) > \theta$ and $\mathscr{T}_* \equiv \mathscr{T}_{\geqslant\theta} \times \mathscr{T}_{<\theta}$ and $\mathscr{T}_{\geqslant\theta}$ is θ^+-complete, $\min(\Theta_1\backslash\theta) > \Pi(\Theta_1 \cap \theta) + \aleph_1$, without loss of generality $\underset{\sim}{\mathscr{A}}$ is a $(\mathbb{P} \times \mathscr{T}_{<\theta})$-name. As $\lambda \geqslant \mathrm{cf}(\lambda) > \theta \geqslant \Pi(\Theta_1 \cap \theta)$ by 2.1(1)(e) for some $\varepsilon < \lambda, \underset{\sim}{\mathscr{A}}$ is a $(\mathbb{P}^{\mathbf{x}_\varepsilon}_{\delta(*)} \times \mathscr{T}_{<\theta})$-name. As we can increase α without loss of generality $\mathrm{cf}(\alpha) = \theta$. Now apply 2.13 recalling clause (c) of $(*)$. $\square_{2.17}$

3. The $\aleph_n$'s and collapsing

A drawback of 2.17 is that $\mathbf{V}$ and $\mathbf{V}^{\mathbb{P}}$ have the same cardinals while the cardinals missing from Sp_χ are ex-large cardinals so weakly inaccessible. In particular it gives no information on chaotic behavior of Sp_χ among the $\aleph_n$'s. This is resolved to a large extent below. However, here we do not improve the consistency strength, also we do not deal here with successor of singulars but deal little with singulars.

So fulfilling the second promise from Section 0 (the first was dealt with in Section 2, i.e. 2.17) the main result of this section is:

Conclusion 3.1. 1) *If $u \subseteq \{1, 2, \dots, n, \dots\}$ and $n \geqslant 1 \Rightarrow n \in u \vee n+1 \in u$ and in* **V** *there are infinitely many measurable cardinals, then for some forcing notion $\mathbb{P}$ in $\mathbf{V}^{\mathbb{P}}$ we have $\aleph_\omega \cap \mathrm{Sp}_\chi = \{\aleph_n \colon n \in u\}$.*

2) *Assume in* **V** *there are infinitely many compact cardinals. Then in part* (1) *we can use any $u \subseteq [1, \omega)$.*

Proof. Straightforward from 3.2, 3.4 below. $\square_{3.1}$

Claim 3.2. *Assume GCH for simplicity, Hypothesis* 2.1 *and $\theta \in \Theta_2 \Rightarrow \theta > \sup(\theta \cap \Theta)$ and $\mathscr{T}_\theta$ is θ-complete for $\theta \in \Theta_1$, $\lambda = \mathrm{cf}(\lambda)$ for simplicity; let $\mathbf{f}$ be a function with domain Θ_2 such that $\theta > \mathbf{f}(\theta) > \sup(\Theta \cap \theta)$, $\mathbf{f}(\theta) > \aleph_1$ is regular (so $\mathbf{f}(\theta)^{<\mathbf{f}(\theta)} = \mathbf{f}(\theta)$) and $\mathbf{f}(\theta) \notin \Theta_2$ and let $\mathbb{Q}$ be the product $\Pi\{\mathrm{Levy}(\mathbf{f}(\theta), <\theta) \colon \theta \in \Theta_2\}$ with Easton support (recall $\mathrm{Levy}(\mathbf{f}(\theta), <\theta)$ is collapsing each $\alpha \in [\mathbf{f}(\theta), \theta)$ to $\mathbf{f}(\theta)$ by approximation of cardinality $< \mathbf{f}(\theta)$).*

Lastly, let $\mathbf{x} = \mathbf{x}_\lambda$, $\delta(*)$ be as in the proof of 2.17. *Then $\mathbb{P} = \mathbb{P}^{\mathbf{x}}_{\delta(*)} \times \mathscr{T} \times \mathbb{Q}$ satisfies:*

(a) *$\mathbb{P}$ is a forcing notion of cardinality λ and $\Vdash_{\mathbb{P}}$ "$2^{\aleph_0} = \lambda$"*
(b) *$\mathscr{T}$ has cardinality $\leqslant \Pi\Theta_1$, and as a forcing notion adds no new sequence of length $< \min(\Theta_1)$ of ordinals, collapses no cardinal, changes no cofinality*
(c) *$\mathbb{P}$ has cardinality $\leqslant \lambda + \Pi\Theta_1$, really $\lambda + |\Pi\mathscr{T}_*| + |\mathbb{Q}|$, collapses no cardinal except those in $\bigcup\{(\mathbf{f}(\theta), \theta) \colon \theta \in \Theta_2\}$, changes no cofinality except that $\mathrm{cf}^{\mathbf{V}}(\delta) = (\mathbf{f}(\theta), \theta) \Rightarrow \mathrm{cf}^{\mathbf{V}[\mathbb{P}]}(\delta) = \mathbf{f}(\theta)$*
(d) *in $\mathbf{V}^{\mathbb{P}}$ we have $\Theta_1 \subseteq \mathrm{Sp}_\chi$, i.e. for every $\theta \in \Theta_1$ there is a non-principal ultrafilter D of character θ*
(e) *in $\mathbf{V}^{\mathbb{P}}$ we have $\Theta_2 \cap \mathrm{Sp}_\chi = \emptyset$.*

Discussion 3.3. 1) We may allow $\mathbf{f}(\theta) = \sup(\Theta \cap \theta)$ when $\sup(\Theta \cap \theta) \notin \Theta_2$.

2) We may like to have successive members of Θ_2, see 3.4; together with 3.3(1) we get full answer for the $\aleph_n$'s.

3) We may in 3.2, if $\lambda = \lambda^{<\kappa}$ demand $\Vdash_{\mathbb{P}}$ "$\mathrm{MA}_{<\kappa}$", for this we need in the inductive choice of the $\mathbf{x}_\varepsilon$'s for $\varepsilon < \lambda$ another case; we do not get $\mathrm{MA}_{\leqslant\kappa}$ as $\mathrm{cf}(\delta(*)) = \kappa$.

4) Similarly to part (3) in 2.17, 3.6, 3.1.

Proof. First, clause (c), on when cardinals and cofinalities are preserved should be clear. Second, note that forcing by $\mathscr{T}_* \times \mathbb{Q}$ adds no new ω-sequence of members of **V** and even preserve "$\mathbb{P}_{\mathbf{x}_\lambda}$ satisfies c.c.c." (and even "satisfies the Knaster condition" and even "being locally $\aleph_1$-centered") all because $\mathscr{T}_* \times \mathbb{Q}$ is $\aleph_1$-complete. So $\mathscr{P}(\mathbb{N})^{\mathbf{V}[\mathbb{P}]}$ and even $({}^{\omega}\mathrm{Ord})^{\mathbf{V}[\mathbb{P}]}$ is the same as the one in $\mathbf{V}[\mathbb{P}^{\mathbf{x}}_{\delta(*)}]$.

Third, note that for every $\theta \in \Theta_1$, in $\mathbf{V}^{\mathscr{T}_*}$ we have a $\mathbb{P}^{\mathbf{x}}_{\delta(*)}$-name $\underset{\sim}{D}_\theta$ of an ultrafilter on $\mathbb{N}$ with $\chi(\underset{\sim}{D}_\theta) = \theta$, so there is a set $\mathbb{D}_\theta$ of $\mathbb{P}^{\mathbf{x}}_{\delta(*)}$-names of reals of cardinality θ, or better a set of representations of such names (see Definition 2.14), which generates $\underset{\sim}{D}_\theta$.

Now $\underset{\sim}{D}_\theta$ has the same properties in $\mathbf{V}^{\mathscr{T}_* \times \mathbb{Q}}$ (see "first" and "second" above) so we have $\theta \in \mathrm{Sp}_\chi^{\mathbf{V}[\mathbb{P}]}$ so $\mathbf{V}^{\mathbb{P}} \models$ "$\Theta_1 \subseteq \mathrm{Sp}_\chi$".

Fourth, the main point, we would like to prove that $\Theta_2 \cap \mathrm{Sp}_\chi = \emptyset$ in $\mathbf{V}^{\mathbb{P}}$.

So toward contradiction assume

$\odot_1$ $\theta \in \Theta_2$ and $(p^*, r^*, q^*) \in \mathbb{P}$ forces "$\underset{\sim}{D}$ is an ultrafilter on $\mathbb{N}$ with $\chi(\underset{\sim}{D}) = \theta$".

Let $\mathbb{Q}_{<\theta}$ be $\{p \in \mathbb{Q} \colon \mathrm{dom}(q) \subseteq \theta\}$ and similarly $\mathbb{Q}_{\leqslant\theta}, \mathbb{Q}_{>\theta}$ so essentially $\mathbb{Q} = \mathbb{Q}_{\leqslant\theta} \times \mathbb{Q}_{>\theta}$ and $\mathbb{Q}_{\leqslant\theta} = \mathbb{Q}_{<\theta} \times \mathbb{Q}_\theta$ where $\mathbb{Q}_\theta = \mathrm{Levy}(\mathbf{f}(\theta), <\theta)$. Similarly $\mathscr{T}_{<\theta} = \{r \in \mathscr{T} \colon \mathrm{dom}(r) \subseteq \theta\}$, etc.

Now

$(*)_1$ $|\mathscr{T}_{<\theta} \times \mathbb{Q}_{<\theta}| < \theta$.

[Why? Recalling $|\mathscr{T}_{<\theta}| \leqslant (\sup(\Theta_1 \cap \theta))^+ \leqslant (\sup(\Theta \cap \theta))^+ \leqslant \mathbf{f}(\theta)^+ < \theta$ by an assumption on $\mathbf{f}$ and $\mathbb{Q}_{<\theta} \leqslant \Pi\{\mathbb{Q}_\partial \colon \partial \in \Theta_2 \cap \theta\}$ has cardinality $\leqslant \sup(\Theta_2 \cap \theta)^+ \leqslant \mathbf{f}(\theta)^+ < \theta$.]

$(*)_2$ there is a sequence $\langle \mathbf{p}_\varepsilon \colon \varepsilon < \theta \rangle$, $\underset{\sim}{\mathbf{p}}_\varepsilon$ a $(\mathscr{T}_* \times \mathbb{Q})$-name of a $\mathbb{P}^{\mathbf{x}}_{\delta(*)}$-representation of a subset $\underset{\sim}{A}_\varepsilon$ of $\mathbb{N}$ such that $(p^*, r^*, q^*) \Vdash_{\mathbb{P}}$ "$\{\underset{\sim}{A}_\varepsilon \colon \varepsilon < \theta\}$ generates $\underset{\sim}{D}$ and $\underset{\sim}{A}_n \cap [0, n) = \emptyset$ and $\chi(\underset{\sim}{D}) = \theta$"

$(*)_3$ without loss of generality $(p^*, r^*, q^*) \in \mathbb{P}' := \mathbb{P}^{\mathbf{x}}_{\delta(*)} \times \mathscr{T}_{<\theta} \times \mathbb{Q}_{\leqslant\theta}$ and $\underset{\sim}{D}$, moreover the sequence $\langle \underset{\sim}{\mathbf{p}}_\varepsilon \colon \varepsilon < \theta \rangle$ are $\mathbb{P}'$-names.

[Why? Because, first, $\mathbb{Q}/\mathbb{Q}_{\leqslant\theta}$ is θ^+-complete as we are assuming $\sigma \in \Theta_2 \backslash \theta^+ \Rightarrow \mathbf{f}(\sigma) > \theta$. Second, recalling $\theta \notin \Theta_1$ as Θ_1, Θ_2 are disjoint, forcing by $\mathscr{T}_{\geqslant\theta} = \mathscr{T}_{>\theta}$ adds no new sequence of length $\leqslant \theta$ of ordinals (by 2.1) and even is θ^+-complete (by the claim assumptions). Third, $\mathscr{T}_{<\theta} \times \mathbb{Q}_{\leqslant\theta}$ has cardinality $\leqslant \theta$.]

$(*)_4$ there are $\langle (r_\varepsilon, q_\varepsilon, \mathbf{q}_\varepsilon, \underset{\sim}{A}'_\varepsilon)\colon \varepsilon < \theta\rangle$ such that:
(a) $r_\varepsilon \in \mathscr{T}_{<\theta}$ and $q_\varepsilon \in \mathbb{Q}_{\leqslant\theta}$
(b) $\mathbf{q}_\varepsilon$ is a canonical representation of a $\mathbb{P}^{\mathbf{x}}_{\delta(*)}$-name of a subset of $\mathbb{N}$
(c) $(p^*, r_\varepsilon, q_\varepsilon)$ belongs to $\mathbb{P}^{\mathbf{x}}_{\delta(*)} \times \mathscr{T}_{<\theta} \times \mathbb{Q}_{\leqslant\theta}$, is above (p^*, r^*, q^*) and forces that $\underset{\sim}{A}_{\mathbf{q}_\varepsilon}, \mathbb{N}\backslash \underset{\sim}{A}_{\mathbf{q}_\varepsilon}$ are $\neq \emptyset$ mod fil$(\{\underset{\sim}{A}_\iota\colon \iota < \varepsilon\})$ and $\{\underset{\sim}{A}_\iota\colon \iota < \varepsilon\}$ is included in this filter and the condition also forces $\underset{\sim}{\mathbf{p}}_\varepsilon$ is $\mathbf{q}_\varepsilon$
(d) $\underset{\sim}{A}'_\varepsilon$ is the $\mathbb{P}_{\mathbf{x}}$-name of a subset of $\mathbb{N}$ represented by $\mathbf{q}_\varepsilon$
(e) for technical reasons $\theta \in \mathrm{dom}(q^*_\varepsilon)$.

[Why? As (p^*, r^*, q^*) forces that $\{\underset{\sim}{A}_\iota\colon \iota < \theta\}$ generates $\underset{\sim}{D}$ but $\underset{\sim}{A}_\varepsilon \notin \mathrm{fil}(\{\underset{\sim}{A}_\zeta\colon \zeta < \varepsilon\})$.]
Easily

$(*)_5$ there are representations $\mathbf{q}'_i (i < \theta)$ of $\mathbb{P}^{\mathbf{x}}_{\delta(*)}$-names $\underset{\sim}{C}_i$ such that
(a) $(p^*, r^*, q^*) \Vdash_{\mathbb{P}'}$ "$\underset{\sim}{\mathbf{p}}_\varepsilon \in \{\mathbf{q}'_i\colon i < \theta\}$" for every $\varepsilon < \theta$
(b) $(p^*, r^*, q^*) \Vdash$ "$\{\underset{\sim}{C}_i\colon i < \theta\}$ includes $\{\underset{\sim}{A}_i\colon i < \theta\}$ and is closed under (the finitary) Boolean operations"
(c) $(p^*, r^*, q^*) \Vdash_{\mathbb{P}^{\mathbf{x}}_{\delta(*)} \times \mathscr{T}_{<\theta} \times \mathbb{Q}_{\leqslant\theta}}$ "$\{\underset{\sim}{C}_i\colon i < \theta\} \cap \underset{\sim}{D}$ generated $\underset{\sim}{D}$ and for some club E of θ, if $\varepsilon < \theta$ then $\{\underset{\sim}{\mathbf{p}}_\zeta\colon \zeta < \varepsilon\}$, $\{\underset{\sim}{C}_i\colon i < \varepsilon\} \cap \underset{\sim}{D}$ generate the same filter"
(d) E is actually a club of θ from $\mathbf{V}$
(e) $\varepsilon \in E \Rightarrow (p^*, r^*, q^*) \Vdash$ "$\{\underset{\sim}{C}_i\colon i < \varepsilon\}$ is closed under the (finitary) Boolean operations", so even p^* forces this (for $\Vdash_{\mathbb{P}_{\delta(*)}[\mathbf{x}_\varepsilon]}$)

$(*)_6$ there are r_*, q_* from $\mathscr{T}_{<\theta}, \mathbb{Q}_{\leqslant\theta}$ respectively and $\mathscr{U} \in E_\theta$ such that
(a) $\varepsilon \in \mathscr{U} \Rightarrow r_\varepsilon = r_* \wedge q_\varepsilon \restriction \theta = q_* \restriction \theta$ so $r^* \leqslant_{\mathscr{T}_{<\theta}} r_\varepsilon$; also $q_* \leqslant_{\mathbb{Q}_{\leqslant\theta}} q_\varepsilon$
(b) $\langle q_\varepsilon(\theta)\colon \varepsilon \in \mathscr{U}\rangle$ is a Δ-system with heart $q_*(\theta) \in \mathbb{Q}_\theta$ and $\mathscr{U} \in E_\theta$
(c) if $\varepsilon_1 < \varepsilon_2, \varepsilon_1 \in \mathscr{U}, \varepsilon_2 \in \mathscr{U}$ <u>then</u> $q_{\varepsilon_1}, q_{\varepsilon_2}$ are compatible[5]
(d) $\mathscr{U} \subseteq E$ where E is from $(*)_5$(d)

[Why? By the proof of Levy$(\mathbf{f}(\theta), <\theta) \models \theta$-c.c.]

$(*)_7$ for $\xi < \zeta < \theta$ let $\underset{\sim}{D}'_{\xi,\zeta}$ be the following $\mathbb{P}^{\mathbf{x}}_{\delta(*)}$-name: it is the filter on $\mathbb{N}$ generated by the family $\{\sigma(\underset{\sim}{C}_{i_0}, \ldots, \underset{\sim}{C}_{i_n-1})$: $\sigma(x_0, \ldots, x_{n-1})$ is a Boolean term and for some $\varepsilon \in \mathscr{U} \cap \zeta\backslash\xi$ we have $\ell < n \Rightarrow i_\ell \in (\xi, \varepsilon)$ and $\underset{\sim}{A}_\varepsilon \subseteq^* \sigma(\underset{\sim}{C}_{i_0}, \ldots, \underset{\sim}{C}_{i_{n-1}})\}$
$(*)_8$ $\Vdash_{\mathbb{P}^{\mathbf{x}}_{\delta(*)}}$ "$\langle \underset{\sim}{D}'_{\xi,\zeta}\colon \zeta \in (\xi, \theta]\rangle$ is increasing continuous for each $\xi < \theta$ and $\langle \underset{\sim}{D}'_{\xi,\zeta}\colon \xi < \zeta\rangle$ is decreasing for each $\zeta < \theta$ and $\emptyset \notin \underset{\sim}{D}'_{\xi,\zeta}$ for $\xi < \zeta < \theta$ and if $\xi < \zeta \in \mathscr{U}$ then $\underset{\sim}{A}_\zeta, \mathbb{N}\backslash \underset{\sim}{A}_\zeta$ are $\neq \emptyset$ mod $\underset{\sim}{D}'_{\xi,\zeta}$".

Recall $\theta < \lambda = \mathrm{cf}(\lambda)$ and so $\langle \mathbb{P}_{\delta(*)}[\mathbf{x}_\varepsilon]\colon \varepsilon < \lambda\rangle$ is $\lessdot$-increasing with union $\mathbb{P}^{\mathbf{x}}_{\delta(*)}$, hence there is $\gamma(*) < \lambda$ of cofinality θ such that for every $\varepsilon < \theta, \mathbf{q}_\varepsilon, \mathbf{q}'_\varepsilon$ are representations of $\mathbb{P}_{\delta(*)}[\mathbf{x}_{\gamma(*)}]$-name so $\underset{\sim}{A}'_\varepsilon, \underset{\sim}{C}_\varepsilon$ are $\mathbb{P}_{\delta(*)}[\mathbf{x}_{\gamma(*)}]$-names and let $\mathbf{j}_{\gamma(*)}$ be the $\mathbf{j}_*$ from 2.13, so $(\mathbf{j}_{\gamma(*)}, \mathbf{x}_{\gamma(*)}, \mathbf{x}_{\gamma(*)+1})$ here stand for $(\mathbf{j}_*, \mathbf{x}, \mathbf{y})$ there.

Recall $\langle \mathbb{P}_{\delta(*)}[\mathbf{x}_\varepsilon]\colon \varepsilon < \lambda\rangle$ is $\lessdot$ increasing and is continuous for ordinals of cofinality $> \aleph_0$. Let $\underset{\sim}{A}'_\theta$ be $\mathbf{j}_{\gamma(*)}(\langle \underset{\sim}{A}'_\varepsilon\colon \varepsilon \in \mathscr{U}\rangle / E_\theta)$, well abusing our notation a little; you may prefer to use $\mathbf{q}_\theta = \mathbf{j}_{\gamma(*)}(\langle \mathbf{q}_\varepsilon\colon \varepsilon < \theta\rangle / E_\theta)$ and $\underset{\sim}{A}'_\theta$ be the $\mathbb{P}_{\delta(*)}[\mathbf{x}_{\gamma(*)+1}]$-name represented by $\mathbf{q}_\theta$.

Now as $(p^*, r^*, q^*) \Vdash$ "$\{\underset{\sim}{C}_i\colon i < \theta\} \cap \underset{\sim}{D}$ generate an ultrafilter on $\mathbb{N}$" and (p^*, r^*, q^*) is below $(p^*, q_{\min(\mathscr{U})}, r_{\min(\mathscr{U})})$ so there is $(p^1, r^1, q^1) \in \mathbb{P}^{\mathbf{x}}_{\alpha(*)} * \mathscr{T}_{<\theta} * \mathbb{Q}_{\leqslant\theta}$ above it, $n \in \mathbb{N}, \varepsilon_0, \ldots, \varepsilon_{n-1} < \theta$, Boolean term $\sigma(x_0, \ldots, x_{n-1})$ and truth value $\mathbf{t}$ such that

$(*)_9$ (p^1, r^1, q^1) forces $\sigma(\underset{\sim}{C}_{\varepsilon_0}, \ldots, \underset{\sim}{C}_{\varepsilon_{n-1}}) \in \underset{\sim}{D}$ and is included in $(\underset{\sim}{A}'_\theta)^{[\mathbf{t}]}$ recalling $A^{[1]} = A, A^{[0]} = \mathbb{N}\backslash A$

<u>hence</u>

$(*)_{10}$ $p^1 \Vdash_{\mathbb{P}_{\delta(*)}[\mathbf{x}]}$ "$\sigma(\underset{\sim}{C}_{\varepsilon_0}, \ldots, \underset{\sim}{C}_{\varepsilon_{n-1}}) \subseteq^* (\underset{\sim}{A}'_\theta)^{[\mathbf{t}]}$".

Let $p^2 \in \mathbb{P}_{\delta(*)}[\mathbf{x}_{\gamma(*)+1}]$ be such that $p^2 \leqslant p^* \in \mathbb{P}_{\delta(*)}[\mathbf{x}_{\gamma(*)+1}] \Rightarrow p^1, p^*$ compatible, so clearly

$(*)_{11}$ $p^2 \Vdash_{\mathbb{P}_{\alpha(*)}[\mathbf{x}_{\gamma(*)+1}]}$ "$\sigma(\underset{\sim}{C}_{\varepsilon_0}, \ldots, \underset{\sim}{C}_{\varepsilon_{n-1}})$ is $\subseteq^* (\underset{\sim}{A}'_\theta)^{[\mathbf{t}]}$".

Let $\langle p^2_\varepsilon\colon \varepsilon < \theta\rangle \in {}^\theta(\mathbb{P}_{\delta(*)}[\mathbf{x}_{\gamma(*)}])$ be such that $\mathbf{j}_{\gamma(*)}(\langle p^2_\varepsilon\colon \varepsilon < \theta\rangle) = p^2$.
Hence

$(*)_{12}$ $\mathscr{U}_1 = \{\zeta \in \mathscr{U}\colon p^2_\zeta \Vdash$ "$\sigma(\underset{\sim}{C}_{\varepsilon_0}, \ldots, \underset{\sim}{C}_{\varepsilon_{n-1}})$ is $\subseteq^* (\underset{\sim}{A}'_\zeta)^{[\mathbf{t}]}$"$\}$ belongs to E_θ.

[5] So even any $< \mathbf{f}(\theta)$ members are.

Without loss of generality $\langle p^2_\zeta\colon \zeta \in \mathscr{U}_1\rangle$ are pairwise compatible hence by Łos theorem for some ζ

$(*)_{13}$ $\zeta \in \mathscr{U}_1$ so $\zeta < \theta$ and p^2, p^2_ζ has a common upper bound $p^3 \in \mathbb{P}_{\mathbf{x}_{\gamma(*)+1}}$, hence p^1, p^3 has a common upper bound $p^4 \in \mathbb{P}^{\mathbf{x}}_{\delta(*)}$.

So recalling q_ζ is from $(*)_4$,

$(*)_{14}$ (p^4, r_*, q_ζ) forces
(a) $\underset{\sim}{A}'_\zeta \in \underset{\sim}{D}$
(b) $\sigma(\underset{\sim}{C}_{\varepsilon_0}, \ldots, \underset{\sim}{C}_{\varepsilon_{n-1}}) \in \underset{\sim}{D}$
(c) $\sigma(\underset{\sim}{C}_{\varepsilon_0}, \ldots, \underset{\sim}{C}_{\varepsilon_{n-1}}) \subseteq^* (\underset{\sim}{A}'_\theta)^{[\mathbf{t}]}, (\underset{\sim}{A}'_\zeta)^{[\mathbf{t}]}$.

Contradiction. $\square_{3.2}$

Claim 3.4. *In* 3.2 (*and* 1.6) *instead of E_θ is θ-complete* (*so θ is measurable*) *we may require that there is $\Theta'_2 \subseteq \Theta_2$ such that*:

(a) $(\Theta'_2, \mathbf{f})$ *are as in* 3.2
(b) *defining $\mathbb{Q}$ we use Θ'_2 if $\theta \in \Theta'_2$ then E_θ is θ-complete*
(c) *if $\sigma \in \Theta_2 \backslash \Theta'_2$ <u>then</u> $\theta = \max(\Theta'_2 \cap \sigma)$ is well defined, $[\theta, \sigma] \cap \Theta_1 = \emptyset$ and E_θ is a uniform θ-complete ultrafilter on σ so θ is a σ-compact cardinal.*

Proof. Similar to 3.2. $\square_{3.4}$

Remark 3.5. The situation is similar for any set $\{\aleph_\alpha\colon \alpha \in u\}$ of successor of regular cardinals.

Claim 3.6. *In* 3.1 *above the sufficient conditions for "$\theta \notin \mathrm{Sp}_\chi$ in $\mathbf{V}^{\mathbb{P}}$" are sufficient also for "$(\forall\mu)(\mathrm{cf}(\mu)) = \theta \Rightarrow (\mu \notin \mathrm{Sp}_\chi)$".*

Proof. The same. $\square_{3.6}$

So we can resolve Problem (6) from Brendle and Shelah [3, §8].

Conclusion 3.7. *If GCH and $\aleph_1 \leqslant \theta < \kappa = \mathrm{cf}(\kappa) < \lambda = \lambda^\kappa$, κ is measurable, <u>then</u> there is a forcing notion $\mathbb{P}$ of cardinality λ collapsing the cardinals in (θ, κ) but no others such that in $\mathbf{V}^{\mathbb{P}}$, for every cardinal $\mu \in (\kappa, \lambda)$ of cofinality κ, we have $\mu \notin \mathrm{Sp}_\chi \wedge \mu = \sup(\mathrm{Sp}_\chi \cap \mu)$.*

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